Cirrito 5.4.1 Composition p156/Q22

22/ $f(x) = \frac{1}{a}x^2$ $\qquad g(x) = \sqrt{2a^2 - x^2}$

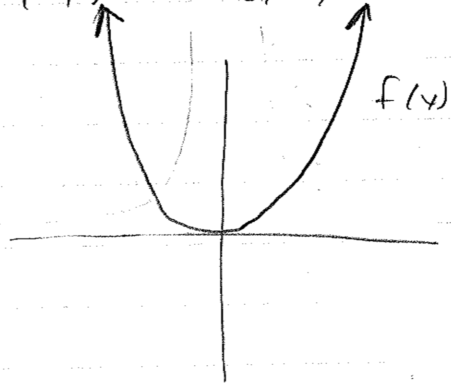
rearrange

$y^2 = 2a^2 - x^2$

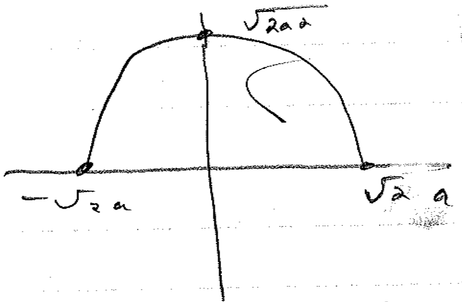
$x^2 + y^2 = 2a^2$

i.e. semi circle

(a)



$f(x) = \frac{1}{a}x^2$

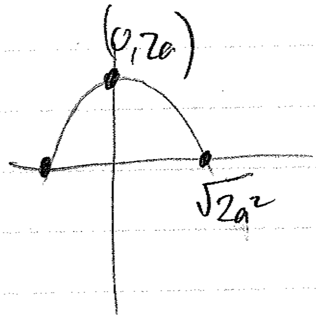


(b) $f \circ g(x)$ exists $\Rightarrow$ check Range of $g(x)$

Range $g(x)$: $\{y \in \mathbb{R} \mid 0 \leq y \leq \sqrt{2a^2}\}$

all of which are valid <u>inputs</u> for $f(x)$

$$
\begin{aligned}
f \circ g(x) &= f\left(\sqrt{2a^2 - x^2}\right) \\
&= \frac{1}{a}\left(\sqrt{2a^2 - x^2}\right)^2 \\
&= \frac{1}{a}\left(2a^2 - x^2\right) \\
&= 2a - \frac{x^2}{a} \qquad \text{or} \qquad \frac{2a^2 - x^2}{a} \\
&= -\frac{1}{a}x^2 + 2a
\end{aligned}
$$



$\Rightarrow$ parabola opens down (0, 2a) vertex

Domain of $f \circ g(x)$ $\Rightarrow$ check range of $g(x)$

$f \circ g(0) = 2a$

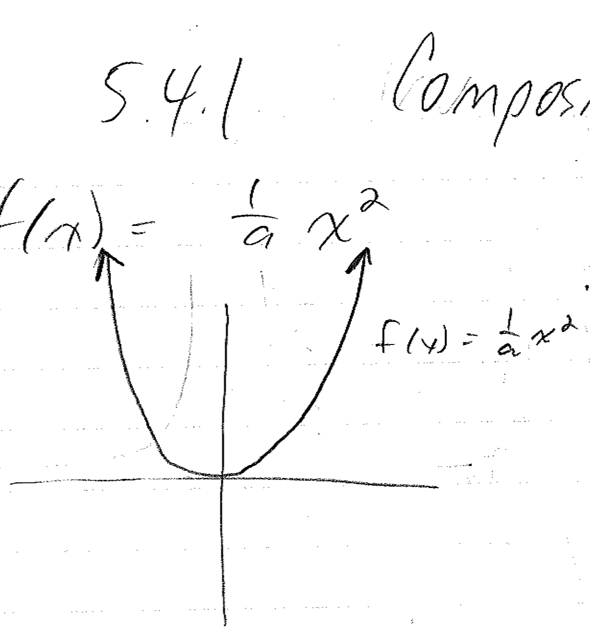
$f \circ g(\sqrt{2a^2}) = -\frac{1}{a}(\sqrt{2a^2})^2 + 2a = -\frac{1}{a}(2a^2) + 2a = 0$

$\therefore$ Domain of $f \circ g(x)$: $\{x \in \mathbb{R} \mid 0 \leq x \leq \sqrt{2a^2}\}$

Cirrito 5.4.1 Composition p156 /Q22

22/ $f(x) = \frac{1}{a}x^2$ $\qquad$ $g(x) = \sqrt{2a^2 - x^2}$

(a)



rearrange

$y^2 = 2a^2 - x^2$

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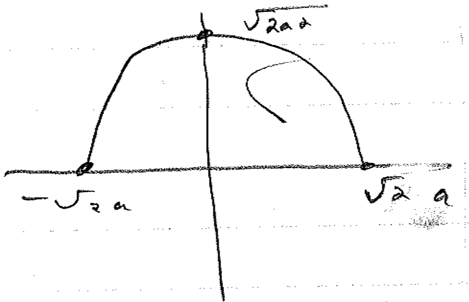
i.e. semi circle



(b) $f \circ g(x)$ exists $\Rightarrow$ check Range of $g(x)$

Range $g(x)$: $\{y \in \mathbb{R} \mid 0 \le y \le \sqrt{2a^2}\}$

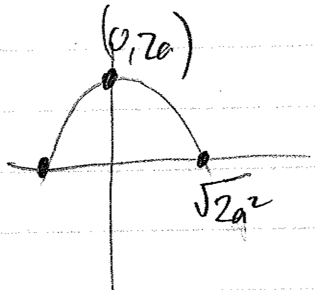
all of which are valid inputs for $f(x)$

$$f \circ g(x) = f\left(\sqrt{2a^2 - x^2}\right)$$

$$= \frac{1}{a}\left(\sqrt{2a^2 - x^2}\right)^2$$

$$= \frac{1}{a}\left(2a^2 - x^2\right)$$

$$= 2a - \frac{x^2}{a} \quad \text{or} \quad \frac{2a^2 - x^2}{a}$$



$$= -\frac{1}{a}x^2 + 2a$$

$\Rightarrow$ parabola opens down $(0, 2a)$ vertex

Domain of $f \circ g(x)$ $\Rightarrow$ check range of $g(x)$

$f \circ g(0) = 2a$

$f \circ g(\sqrt{2a^2}) = -\frac{1}{a}(\sqrt{2a^2})^2 + 2a = -\frac{1}{a}(2a^2) + 2a = 0$

$\therefore$ Domain of $f \circ g(x)$: $\{x \in \mathbb{R} \mid 0 \le x \le \sqrt{2a^2}\}$

Cirrito 5.4.1 p156 Q22

22c $g \circ f(x) = g\left(\frac{1}{a}x^2\right)$

check range of $f(x) = \frac{1}{a}x^2 \Rightarrow y \geq 0$

check domain of $g(x) = \sqrt{2a^2 - x^2}$

$$2a^2 - x^2 \geq 0$$
$$2a^2 \geq x^2$$
$$\sqrt{2}|a| \geq |x|$$

so $-\sqrt{2}a \leq x \leq \sqrt{2}a$

$$g\left(\tfrac{1}{a}x^2\right) = \sqrt{2a^2 - \left(\tfrac{1}{a}x^2\right)^2}$$
$$= \sqrt{2a^2 - \tfrac{1}{a^2}x^4}$$
$$= \sqrt{\frac{2a^4 - x^4}{a^2}}$$
$$= \frac{1}{a}\sqrt{2a^4 - x^4}$$

Domain of composite $2a^4 - x^4 \geq 0$

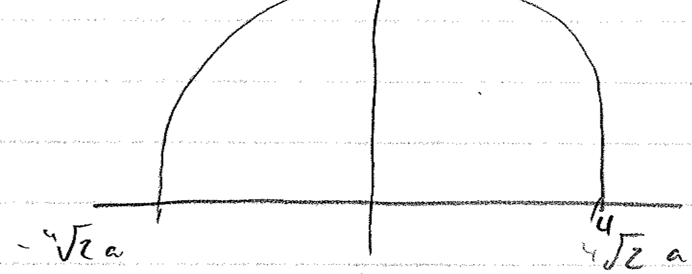
$$2a^4 \geq x^4$$
$$\sqrt[4]{2}|a| \geq |x|$$

so

$\{x \in \mathbb{R} \mid -\sqrt[4]{2}a \leq x \leq \sqrt[4]{2}a\}$

$\frac{1}{a}(\sqrt{2a^2}) \Rightarrow (0, \sqrt{2}a)$

Sketch

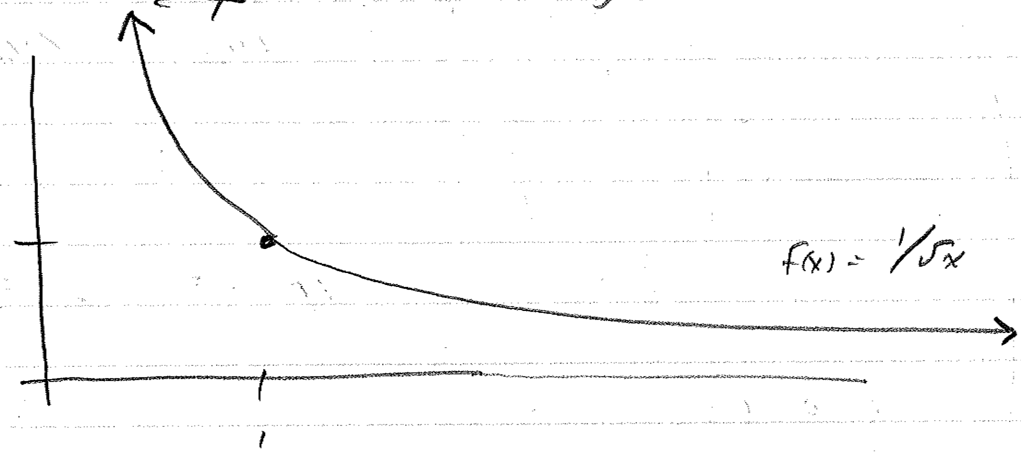


Range $\{y \in \mathbb{R} \mid 0 \leq y \leq \sqrt{2}a\}$

Cirrito 5.4.1 p156 Q19

19 (piecewise fn) $f(x) = \begin{cases} \frac{1}{x^2} & 0 < x \le 1 \\ \frac{1}{\sqrt{x}} & x > 1 \end{cases}$

all output values ⇒ $y > 0$



$f(0)$ = no value
$f(1) = \frac{1}{1^2} = 1$
$f(4) = 1/2$
$f(9) = 1/3$ etc.
$f(16) = 1/4$
decreases slowly

$f(1/2) = 1/(1/2)^2 = 4$
$f(1/3) = 9$ etc.
$f(\frac{1}{4}) = 16$
increases rapidly

(b) $f \circ f(x)$ ⇒ Range of $f(x)$ ⇒ $y > 0$
which matches Domain of $f(x)$ $x > 0$

$f\left(\frac{1}{x^2} \quad 0 < x \le 1\right)$ ⇒ produces $\frac{1}{x^2}$ which is > 0
$\therefore f(\frac{1}{x^2}) = \frac{1}{\sqrt{\frac{1}{x^2}}} =$

$f \circ f(1) = f(f(1)) = f(1) = 1$
$f \circ f(4) = f(f(4)) = f(\frac{1}{2}) = 4$
$f \circ f(9) = f(f(9)) = f(\frac{1}{3}) = 9$
$f \circ f(\frac{1}{4}) = f(f(\frac{1}{4})) = f(16) = \frac{1}{4}$
so..... $f \circ f(x) = f(f x) = x$

Range : $\{y \in \mathbb{R} \mid y > 0\}$ as D: $x > 0$

$f \circ f(x)$ algebraically

(i) if $0 \le x \le 1$ $\quad f(x) = 1/x^2$. but $1/x^2 > 0$ if $0 < x \le 1$

$\therefore f(1/x^2) = 1/\sqrt{1/x^2} = 1/(1/x) = x$

(ii) if $x > 1$ $\quad f(x) = 1/\sqrt{x}$ but $1/\sqrt{x} < 0$ if $x \ge 1$

$\therefore f(1/\sqrt{x}) = 1/\left(\frac{1}{\sqrt{x}}\right)^2 = 1/(1/x) = x$

$\therefore f \circ f(x) = x$