(A) Lesson Context

BIG PICTURE of this UNIT:	• How do algebraically & graphically work with growth and decay applications? • What are logarithms and how do we invert or undo an exponential function? • How do we work with simple algebraic and graphic situations involving the use of logarithms (or inversing exponentials?)		
CONTEXT of this LESSON:	Where we've been We have seen algebra skills and applications related to the parent exponential function $f(x) = AB^x$	Where we are What is the difference between discrete changes and continuous changes? How does that change our exponential function?	Where we are heading How do work with the mathematically model $f(x) = AB^{k(x+c)} + d$?

(B) Lesson Objectives:

a. Review KEY algebra skills associated with Compounding Interest
b. Investigte what happens when the compounding conditions are changed from DISCRETE to CONTINUOUS
c. Introduce and work with the NEW Exponential Function in the form of $f(x) = Ae^x$

(C) EXPLORATION #1: Doubling your Money?

GIVEN: the formula for working with compound interest → $A = P\left(1 + \frac{i}{n}\right)^{nt}$, determine the value after 1 year of a $1000 investment invested at 100% pa under the following compounding conditions:

(a) 100% pa compounded annually	
(b) 100% pa compounded semi-annually	
(c) 100% pa compounded quarterly	
(d) 100% pa compounded daily	
(e) 100% pa compounded hourly	
(f) 100% pa compounded every minute	
(g) 100% pa compounded every second	
(h) 100% pa compounded n times per year	

FINAL QUESTION? → BY WHAT **RATIO** HAS YOUR MONEY INCREASED IN VALUE?

(D) Exploration #2 – Continuous Compounding

GIVEN: the formula for working with compound interest ➔ $A = P\left(1 + \frac{i}{n}\right)^{nt}$, determine the value after 2 years of a $1000 investment invested at 10% pa under the following compounding conditions:

(a) 10% pa compounded annually	
(b) 10% pa compounded semi-annually	
(c) 10% pa compounded quarterly	
(d) 10% pa compounded daily	
(e) 10% pa compounded hourly	
(f) 10% pa compounded every minute	
(g) 10% pa compounded every second	
(h) 10% pa compounded n times per year	

QUESTION ➔ Can we GENERALIZE this process of compounding CONTINUOUSLY?

(E) Continuous Compounding: The Magic behind the Math

(B) Introducing Base *e* – CASE #3

- So we have the expression $1000(1 + 0.1/n)^{(2xn)}$
- Now what happens as we increase the number of times we compound per annum ⇒ i.e. n →∞ ?? (that is … come to the point of **compounding continuously**)
- So we get the idea of an "end behaviour again:

$$1000 \times \left(1 + \frac{0.1}{n}\right)^{(2 \times n)} \quad \text{as } n \to \infty$$

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(B) Introducing Base e – CASE #3

- Now let's rearrange our function
- use a simple substitution ➔ let 0.1/n = 1/x
- Therefore, 0.1x = n ➔ so then $1000\times\left(1+\frac{0.1}{n}\right)^{(2\times n)}$ as $n\to\infty$

becomes $1000\times\left(1+\frac{1}{x}\right)^{(x\times 0.1\times 2)}$ as $x\to\infty$

- Which simplifies to $1000\times\left(\left(1+\frac{1}{x}\right)^{x}\right)^{0.1\times 2}$

(B) Introducing Base e – CASE #3

- So we see a special "end behaviour" occurring:

$$1000\times\left(\left(1+\frac{1}{x}\right)^{x}\right)^{0.1\times 2} \quad \text{as } x\to\infty$$

- We see again that , $e=\left(1+\frac{1}{x}\right)^{x}$ as $x\to\infty$
- where *e* is the natural base of the exponential function

(B) Introducing Base *e* – CASE #3

- So our original formula $1000\times\left(\left(1+\frac{1}{x}\right)^{x}\right)^{0.1\times 2}$ as $x\to\infty$ now becomes A = 1000$e^{0.1\times 2}$ where the 0.1 was the interest rate, 2 was the length of the investment (2 years) and $1000 was the original investment (so A = Pe^{rt}) ➔ so our value becomes $1221.40
- And our general equation can be written as A = Pe^{rt} where P is the original amount, r is the annual growth rate and t is the length of time in years

The main idea behind the MAGIC is that we get a SPECIAL BASE: $\left(1+\frac{1}{x}\right)^{x}$ which takes on the value of

$e = 2.71828 = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}$, so that our "growth/decay" formulas are now written as $A(t) = Pe^{rt}$

(F) <u>Working With Graphs Involving the base, e</u>

Use your TI-84/DESMOS to generate graphs of the following functions and then SKETCH them into your notes

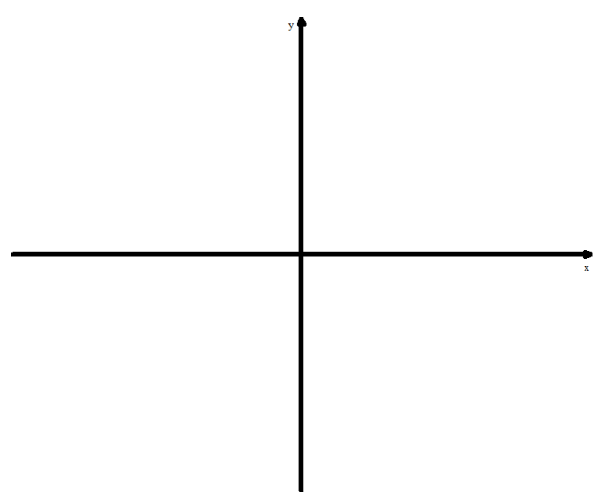
Graph Set A

Graph on one set of axes these three functions: Clearly label the equations on the graphs

$y = 2^x$

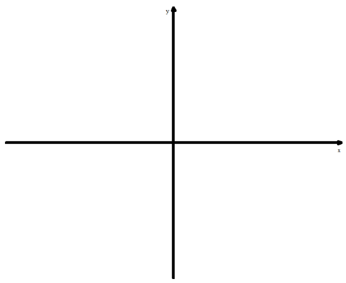
$y = 3^x$

$y = e^x$

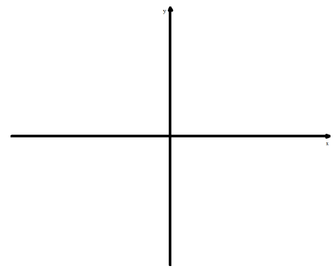


Graph Set B

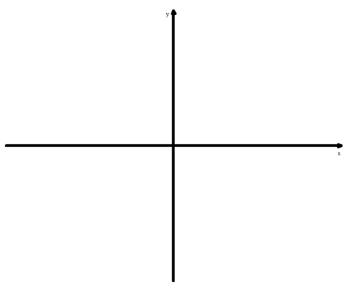
$y = e^{x}$



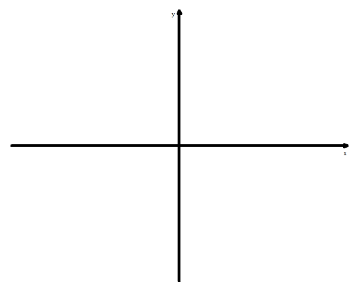
$y = e^{-x}$



$y = -e^{x}$

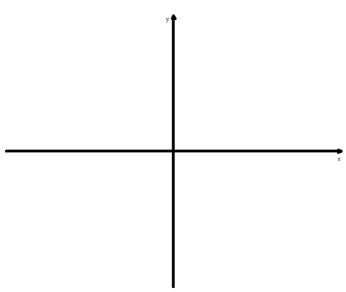


$y = -e^{-x}$

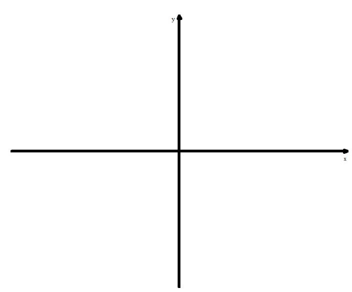


Graph Set C

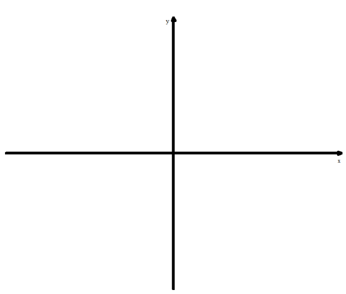
$y = e^{x} + 5$



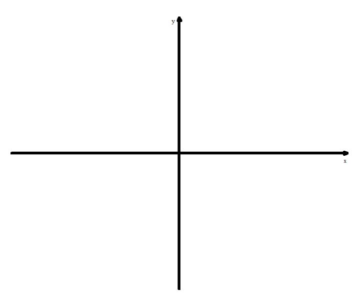
$y = e^{x} - 5$



$y = 5 - e^{x}$



$y = 5 - e^{-x}$

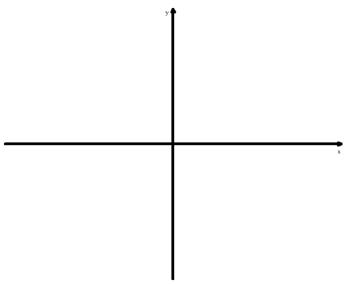


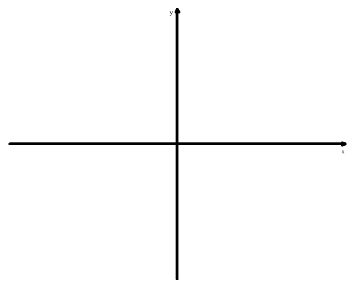
Graph Set D (EXTENSION)

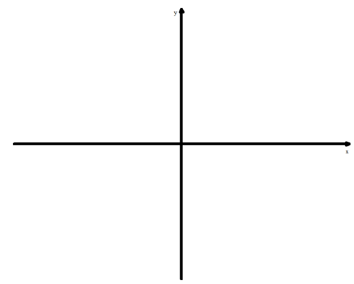
$y = e^{-x^2}$

$y = \dfrac{1}{1+e^{-x}}$

$y = e^{x} + e^{-x}$







(G)<u>Applications</u>

Example #1: I invest $10,000 in a funding yielding 12% p.a. compounded continuously.

(a) Find the value of the investment after 5 years.
(b) How long does it take for the investment to triple in value?

Example #2: The population of the USA can be modeled by the eqn $P(t) = 227e^{0.0093t}$, where P is population in millions and t is time in years since 1980

(a) What is the annual growth rate?
(b) What is the predicted population in 2015?
(c) What assumptions are being made in question (b)?
(d) When will the population reach 500 million?

Example #3: A certain bacteria grows according to the formula $A(t) = 5000e^{0.4055t}$, where t is time in hours.

(a) What will the population be in 8 hours
(b) When will the population reach 1,000,000

Example #4: The function $P(t) = 1 - e^{-0.0479t}$ gives the percentage of the population that has seen a new TV show *t* weeks after it goes on the air.

(a) What percentage of people have seen the show after 24 weeks?
(b) Approximately, when will 90% of the people have seen the show?
(c) What happens to P(t) as t gets infinitely large? Why? Is this reasonable?

Example #5: The number of bacteria in a culture is given by the function $n(t) = 10e^{0.22t}$

(a) What is the relative rate of growth of this bacterium population? Express your answer as a percentage.
(b) What is the initial population of the culture (at t = 0)?
(c) How many bacteria will be in the culture at time t = 15?
(d) What is the doubling time for this bacterial population?

Example #6: A colony of bacteria exhibits continuous growth and doubles every hour. The initial population in a sample of this bacteria is 36.

(a) Determine the number of bacteria after 8 hours.
(b) Determine an exponential model for *N*, the number of bacteria after *t* hours i.e. N(t) = ???
(c) Determine the growth rate PER HOUR.

Example #7: The population of a town is continuously changing. The population of a small town appears to be increasing exponentially. Town planners need a model for predicting the future population. In the year 2000, the population was 35,000, while in the year 2010, the population grew to 57,010.

(a) PREDICT: What will be the town's population in 2030?
(b) Create an **exponential** algebraic model for the town's population growth.
(c) Check your population model by using the fact that the town's population was 72,825 in 2015.
(d) CALCULATE: What will be the town's population in 2030?

Example #8: Three years ago, the fish population in Loon Lake was 2500. Due to the effects of acid rain, there are now about 1950 fish in the lake. Assume that the decline of the fish population is exponential. Find the predicted fish population 5 years from now.

Example #9: What is the average annual rate of inflation if a loaf of bread cost $1.19 in 1991 but costs $1.50 in 2001?