6 a i Interest of 12.5% is added for the year.

Amount owing is $20\,000 \times \frac{112.5}{100} = 20\,000 \times 1.125$

Payment of $\$k$ is made.

Amount owing is $(20\,000 \times 1.125) - k$

ii Amount owing at end of second year is

$((20\,000 \times 1.125) - k) \times 1.125 - k$

iii $(20\,000 \times 1.125) \times 1.125 - 1.125k - k = 17\,131.25$

$\therefore\ 25\,312.5 - 2.125k = 17\,131.25$

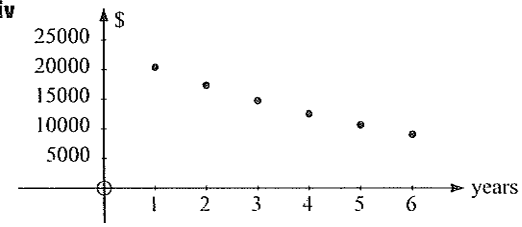
$\therefore\ k = 3850$

b i Percentage decrease $= \dfrac{24\,000 - 20\,400}{24\,000} \times 100\% = 15\%$

ii Value at the end of the 2nd year $= \$20\,400 \times 0.85 = \$17\,340$

iii Value after n years, $t_n = \$24\,000 \times (0.85)^n$

iv



7 a i first month $250 + (5000 \times 1.2\%) = \$310$

ii second month $250 + (4800 \times 1.2\%) = \$307$

iii third month $250 + (4600 \times 1.2\%) = \$304$

b First term is 310. Common difference is -3.

Value of nth payment is $310 + (n-1) \times (-3) = \$(313 - 3n)$

c When $n = 10$, $313 - 3(10) = \$283$

d $\dfrac{5000}{250} = 20$ monthly payments

e $S_{20} = \frac{20}{2}(2 \times 310 + (20-1) \times -3) = \$5630$

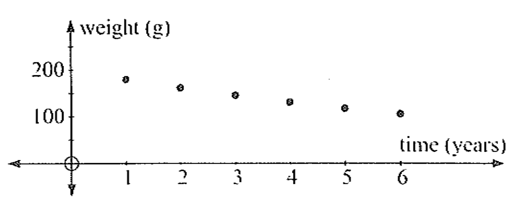
8 a i Weight at start of second year $= 200 \times 0.9 = 180$ g

Weight at start of third year $= 180 \times 0.9 = 162$ g

ii Common ration is 0.9

iii Weight at start of sixth year $= 200 \times 0.9^5 = 118.098$ g

iv



v As $200 \times 0.9^{n-1} = 20$ then $0.9^{n-1} = \frac{20}{200}$

$\therefore\ n \approx 22.9$ {using a gdc}

The material will weigh less than 20 g at the start of the 23rd year.

b Amount of radioactive material at the beginning of the nth year is $120r^{n-1}$.

At the end of the 6th year, or the beginning of the 7th year, the amount is $120 \times r^{7-1} = 49.152$

$\therefore\ r^6 = \dfrac{49.152}{120}$

$\therefore\ r = \left(\dfrac{49.152}{120}\right)^{\frac{1}{6}} = 0.862$

Annual decrease is $1 - 0.862 = 0.138 = 13.8\%$.

SOLUTIONS TO TOPIC 3 SETS, LOGIC AND PROBABILITY

Short questions

1 a $A = \{4, 8, 12, 16\}$, $B = \{2, 4, 6, 8\}$

$A \cup B = \{2, 4, 6, 8, 12, 16\}$

b $A \cap B = \{4, 8\}$

c $C = \{4, 8, 12\}$ or any subset of A that contains 3 elements.

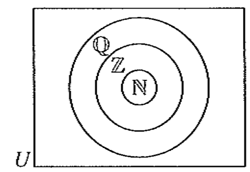
2 a i -6 (or any integer)

ii $3\frac{7}{10}$ (or any number not an integer)

iii 29 (or any positive integer)

iv $\mathbb{Q}' \cap \mathbb{Z} = \varnothing$ and so there are no elements in $\mathbb{Q}' \cap \mathbb{Z}$.

b

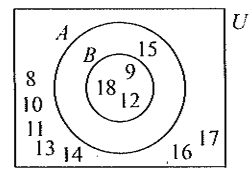


3 $A = \{9, 12, 15, 18\}$, $B = \{9, 12, 18\}$

a $A \cap B = \{9, 12, 18\}$

b $A' = \{8, 10, 11, 13, 14, 16, 17\}$

c



d i $B \subset A$ is true

ii $A' \cap B' = \{8, 10, 11, 13, 14, 16, 17\}$

so $n(A' \cap B') = 7$ is true.

iii Since $B \subset A$, $A \cup B = A$ is true.

4 $F = \{1, 2, 3, 4, 6, 8, 12\}$

$P = \{2, 3, 5, 7, 11, 13, 17, 19\}$

a $n(F) = 7$

b $P \cap F = \{2, 3\}$

c $P \cup F = \{1, 2, 3, 4, 5, 6, 7, 8, 11, 12, 13, 17, 19\}$

d $P' \cap F = \{1, 4, 6, 8, 12\}$

5 $A = \{2, 3, 4, 6, 8, 12, 16, 24, 48\}$

$B = \{6, 12, 18, 24, 30, 36, 42, 48\}$

$C = \{8, 16, 24, 32, 40, 48\}$

a $A \cap B \cap C = \{24, 48\}$

b $n(B) = 8$

c $A' \cap B = \{18, 30, 36, 42\}$

6 a Since b is a prime number less than 10, $b = 2$.

b Q is the set of even numbers less than 10.

c $a = 9$

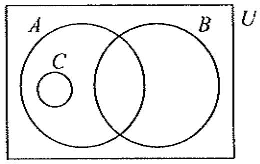
7 a i $A \cap B = \varnothing$ is false as A and B have common elements.

ii $A \cap C' = C$ is true as C lies entirely within A.

iii $B \subset A'$ is false as A and B have common elements.

iv $C \subset (A \cap B)$ is false as C has no elements in common with $A \cap B$.

b The shaded region is $A' \cap B$.



8 a $n(S \cap B \cap G) = 8$, and so 8 members play all three sports.

b $30 + 12$ play soccer but not basketball,
$20 + 9$ play basketball but not soccer and so
$(30 + 12) + (20 + 9) = 71$ play soccer or basketball but not both.

c 14 only play golf and no other sport.

d $30 + 12 + 14 = 56$ do not play basketball.

e $12 + 6 + 9 + 8 = 35$ play two or three sports.

f $30 + 6 = 36$ play soccer but do not play golf.

9 a $p \Rightarrow q$: If the sun is shining then I take the dog for a walk.

b $\neg p \vee q$: The sun is not shining or I take the dog for a walk.

c The converse of $p \Rightarrow q$ is $q \Rightarrow p$: If I take the dog for a walk then the sun is shining.

d $\neg q \Rightarrow \neg p$

e $\neg p \Rightarrow \neg q$ is the contrapositive of $p \Rightarrow q$.

10 a i $\neg q \Rightarrow \neg r$

ii $\neg p \Rightarrow \neg r \wedge \neg q$ or $\neg p \Rightarrow \neg(r \vee q)$

b $\neg r \Rightarrow \neg(p \vee q)$: If Peter is not a student then he has neither black hair nor plays basketball.

11 a i $\neg p \wedge \neg q$: Wilson is not an active dog and Wilson does not dig holes.

ii $p \veebar q$: Wilson is an active dog or Wilson digs holes but not both.

iii $\neg p \Rightarrow q$: If Wilson is not an active dog then Wilson digs holes.

b

p	q	$\neg p$	$\neg p \Rightarrow q$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

(Neither a contradiction nor a tautology.)

12 a i Inverse is $\neg p \Rightarrow \neg q$: If I do not like the beach then I do not live near the sea.

ii Contrapositive is $\neg q \Rightarrow \neg p$: If I do not live near the sea then I do not like the beach.

b Converse is $q \Rightarrow p$.

13

p	q	$\neg q$	$p \Rightarrow \neg q$	$\neg p$	$\neg p \Rightarrow q$	$(p \Rightarrow \neg q) \vee (\neg p \Rightarrow q)$
T	T	F	F	F	T	T
T	F	T	T	F	T	T
F	T	F	T	T	T	T
F	F	T	T	T	F	T

This is a tautology.

14 a i $(\neg r \wedge \neg q) \vee p$: I do not buy an icecream and I have no money or the weather is hot.

ii $(p \wedge q) \Rightarrow r$: If the weather is hot and I have money then I buy an icecream.

iii $(\neg p \vee \neg q) \Rightarrow \neg r$: If the weather is not hot or I have no money then I do not buy an icecream.

b The contrapositive of $p \Rightarrow r$ is $\neg r \Rightarrow \neg p$: If I do not buy an icecream then the weather is not hot.

15 a $\neg p \Rightarrow \neg q$: If Molly does not have a DVD player then Molly does not like watching Movies.

b

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

c

p	q	$\neg p$	$\neg q$	$\neg p \Rightarrow \neg q$
T	T	F	F	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

c As the final column is the same for both propositions then $q \Rightarrow p$ and $\neg p \Rightarrow \neg q$ are logically equivalent.

16 a

p	q	$p \wedge q$	$(p \wedge q) \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

b As the final column only has T values, this is a tautology.

17 a $\neg q$: I do not visit the doctor.

b i $\neg p \Rightarrow q$: If I do not eat an apple every day then I visit the doctor.

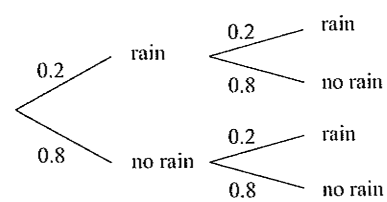
ii $(\neg p \vee q) \Rightarrow q$: If I do not eat an apple every day or I visit the doctor then I visit the doctor.

c

p	q	$\neg p$	$\neg p \vee q$	$(\neg p \vee q) \Rightarrow q$
T	T	F	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	F

As the last column contains some F values and some T values, this is neither a tautology nor a contradiction.

18 a



b i P(raining two days in a row) $= 0.2 \times 0.2$
$= 0.04$

ii P(raining on one day only)
$= \text{P((R and NR) or (NR and R))}$
$= \text{P(R and NR)} + \text{P(NR and R)}$
$= 0.2 \times 0.8 + 0.8 \times 0.2$
$= 0.16 + 0.16$
$= 0.32$

19 a i $A = \{2, 3, 5, 6\}$ and $n(A) = 4$

ii $A \cap B = \{3, 6\}$ and $n(A \cap B) = 2$

b i $A \cup B \cup C = \{2, 3, 4, 5, 6, 7, 8\}$

ii $(A' \cap B) \cup C = \{4, 5, 6, 7, 8\}$

20 a $n(P) = 4 + 1 + 2 + 4 = 11$

b $n(M \cup D) - n(M \cap D) = 8 + 1 + 2 + 6 = 17$

c $n(P \cap M') = 4 + 2 = 6$

d $n(D') = 8 + 1 + 4 + 3 = 16$

$\neg p$:
her is not hot.

DVD player then

$\neg q$	$\neg p \Rightarrow \neg q$
F	T
T	T
F	F
T	T

propositions then
uivalent.

is a tautology.

very day then I

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tradiction.

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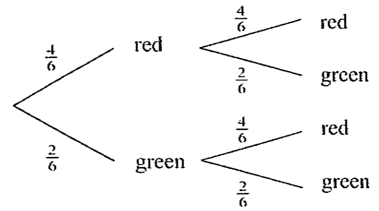
ain

0.2

R))
R)

= 17

21



a P(two reds) $= \frac{2}{3} \times \frac{2}{3} = \frac{4}{9}$

b P(two greens) $= \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$

c P(one of each) $= \frac{2}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{3} = \frac{4}{9}$

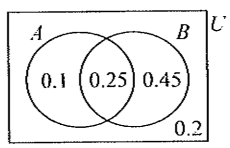
22 a $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\therefore\ 0.8 = 0.35 + 0.7 - P(A \cap B)$

$\therefore\ P(A \cap B) = 1.05 - 0.8$

$\therefore\ P(A \cap B) = 0.25$

b



c $P(A' \cap B') = 0.2$

d If A and B are independent then

$P(A \cap B) = P(A) \times P(B)$

$P(A) \times P(B) = 0.35 \times 0.7$
$= 0.245$

but $P(A \cap B) = 0.25$

$\therefore$ events A and B are not independent.

23 a $c = 1 - (0.3 + 0.4 + 0.2)$
$= 0.1$

b **i** $P(A \cup B) = 0.3 + 0.4 - 0.2 = 0.9$

ii $P(A \cap B) = 0.4$

iii $P(A' \cap B) = 0.2$

iv $P(A \mid B) = \frac{0.4}{0.6} = \frac{2}{3}$

v $P(B \mid A) = \frac{0.4}{0.7} = \frac{4}{7}$

24 a Number in class $= 7 + 9 + 3 + 4$
$= 23$

b $n(C \cap D) = 9$

c $n(D) - n(D \cap C) = 7$

d $n(C \cup D) = 7 + 9 + 3 = 19$

e $n((C \cup D)') = 4$

25 a **i** P(dark or hard) $= \frac{58}{90}$ $(\frac{29}{45}, 0.644)$

ii P(light | soft) $= \frac{18}{28}$ $(\frac{9}{14}, 0.643)$

b **i** P(mm) $= \frac{38}{90} \times \frac{37}{89} = \frac{1406}{8010}$ $(\frac{703}{4005}, 0.176)$

ii P(at least 1 soft | light)

= P(S | L) + P(L | S) + P(both S | L)

$= \left(\frac{18}{90} \times \frac{72}{89}\right) + \left(\frac{72}{90} \times \frac{18}{89}\right) + \left(\frac{18}{90} \times \frac{17}{89}\right)$

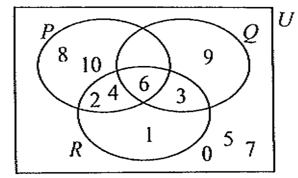
$= \frac{2898}{8010}$ $(\frac{161}{445}$ or $0.362)$

Long questions

1 a **i** $P = \{2, 4, 6, 8, 10\}$ **ii** $Q = \{3, 6, 9\}$

iii $R = \{1, 2, 3, 4, 6\}$ **iv** $P \cap Q \cap R = \{6\}$

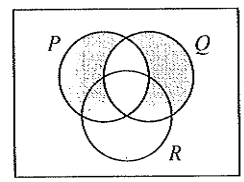
b **i** and **ii**



c **i** $P \cup Q$: The elements that are in set P or Q or both.

ii $P' \cap Q' \cap R'$: The elements that are not in P nor Q nor R.

d **i**



ii

p	q	r	$p \wedge r$	$p \veebar q$	$(p \wedge r) \Rightarrow (p \veebar q)$
T	T	T	T	F	F
T	T	F	F	F	T
T	F	T	T	T	T
T	F	F	F	T	T
F	T	T	F	T	T
F	T	F	F	T	T
F	F	T	F	F	T
F	F	F	F	F	T

iii An element of U for which $(p \wedge r) \Rightarrow (p \veebar q)$ is true is 2 which corresponds to the third entry in the table.

2 $\neg(p \vee q) \Rightarrow \neg p \wedge \neg q$

a **i**

p	q	$\neg p$	$\neg q$	$p \vee q$	$\neg(p \vee q)$	$\neg p \wedge \neg q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

$\neg(p \vee q) \Rightarrow \neg p \wedge \neg q$
T
T
T
T

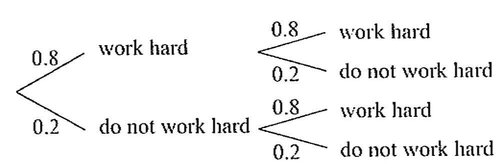
ii This is a tautology.

b **i** $p \wedge q$: I work hard and I get a promotion.

ii Inverse $(\neg p \Rightarrow \neg q)$:

If I do not work hard then I do not get a promotion.

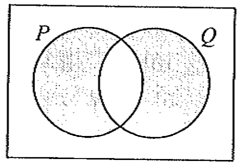
c **i**



ii **a** P(both work hard) $= 0.8 \times 0.8$
$= 0.64$

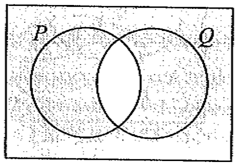
b P(only one works hard) $= 0.8 \times 0.2 + 0.2 \times 0.8$
$= 0.16 + 0.16$
$= 0.32$

3 a i



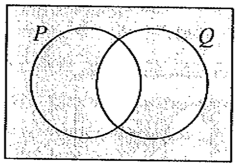
$p \veebar q$

ii



$\neg(p \wedge q)$

iii



$\neg p \vee \neg q$

b Since **a ii** and **a iii** have the same areas shaded, $\neg(p \wedge q)$ and $\neg p \vee \neg q$ are logically equivalent.

c $\neg(p \wedge q) \Rightarrow \neg q$:

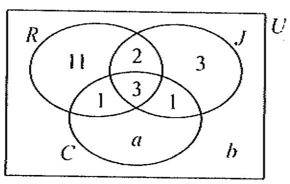
If the bush does not have thorns nor is it a rose bush then it is not a rose bush.

d

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg q$	$\neg(p \wedge q) \Rightarrow \neg q$
T	T	T	F	F	T
T	F	F	T	T	T
F	T	F	T	F	F
F	F	F	T	T	T

The result is neither a tautology nor a contradiction.

4 a



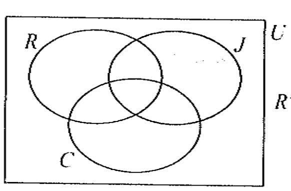
i Since 7 students like Classical music
$n(a) = 7 - (1 + 3 + 1) = 2$

ii The region containing 11 students are those who only like Rock music.

iii Since there are 30 senior students,
$n(b) = 30 - (11 + 2 + 3 + 1 + 3 + 1 + 2)$
$= 30 - 23$
$= 7$

iv Region b consists of the students who do not like any of the music.

b $R' \cap J$



$R' \cap J$

c **i** P(likes all 3 types) $= \frac{3}{30} = \frac{1}{10}$

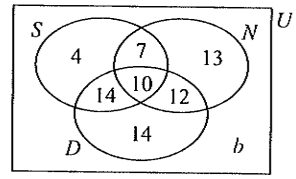
ii P(likes only Classical) $= \frac{2}{30} = \frac{1}{15}$

iii $P(R \mid J) = \frac{5}{9}$

d **i** P(both like Rock music only) $= \frac{11}{30} \times \frac{10}{29} = \frac{110}{870}$ (0.126)

ii P(one likes Rock only and the other likes all 3 types)
$= \left(\frac{11}{30} \times \frac{3}{29}\right) + \left(\frac{3}{30} \times \frac{11}{29}\right)$
$= \frac{66}{870}$ (or 0.0759)

5 a **i** and **ii**



iii $4 + 7 + 10 + 14 + 13 + 12 + 14 = 74$
$\therefore$ $80 - 74 = 6$ students watched none.

b **i** P(watched only Drama) $= \frac{14}{80} = \frac{7}{40}$

ii $P(S \mid N) = \frac{17}{42}$

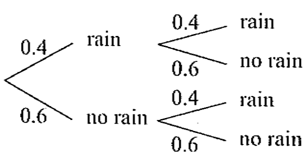
c **i** $p \Rightarrow q$: If you watch sport on television then you like sport.

ii $q \Rightarrow p$: If you like sport then you watch sport on television.

iii $\neg q \Rightarrow \neg p$

iv $\neg q \Rightarrow \neg p$ is the contrapositive of $p \Rightarrow q$.

6 a



b **i** P(Rain on both days) $= 0.4 \times 0.4$
$= 0.16$

ii P(No rain on one day) $= 0.4 \times 0.6 + 0.6 \times 0.4$
$= 0.24 + 0.24$
$= 0.48$

c P(Fine on 5 consecutive days)
$= 0.6 \times 0.6 \times 0.6 \times 0.6 \times 0.6$
$= 0.0778$

d **i** $p \Rightarrow q$: If it is raining then I wear my raincoat.

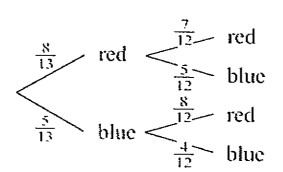
ii $\neg q \Rightarrow \neg p$: If I do not wear my raincoat then it is not raining.

e **i**

p	q	$p \wedge q$	$\neg(p \wedge q)$	$\neg p$	$\neg(p \wedge q) \Rightarrow \neg p$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	F	T	T	T
F	F	F	T	T	T

ii The argument is false when it is raining and I do not wear my raincoat.

7 a **i**



ii a P(two marbles same colour)
$= \left(\frac{8}{13} \times \frac{7}{12}\right) + \left(\frac{5}{13} \times \frac{4}{12}\right)$
$= \frac{76}{132}$
$= \frac{19}{33}$ (0.576)

$= \frac{110}{870}$ (0.126)

likes all 3 types)

U

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ion then you like

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$\Rightarrow q$.

0.6×0.4

y raincoat.

oat then it is not

$(p \wedge q) \Rightarrow \neg p$
T
F
T
T

ng and I do not

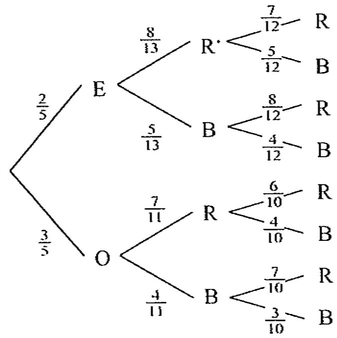
b P(at least one blue marble) $= 1 -$ P(no blue)

$= 1 -$ P(two red)

$= \frac{76}{132}$

$= \frac{19}{33}$ (0.576)

b



i P(even number on the spinner) $= \frac{2}{5}$

ii P(two blue marbles) = P(EBB or OBB)

= P(EBB) + P(OBB)

$= \frac{2}{5} \times \frac{5}{13} \times \frac{4}{12} + \frac{3}{5} \times \frac{4}{11} \times \frac{3}{10}$

≈ 0.117

8 a i P(two chocolates) $= \frac{4}{11}$

ii Since $\frac{3}{10} + \frac{a}{b} = 1$,

then $a = 7$, $b = 10$ is one solution.

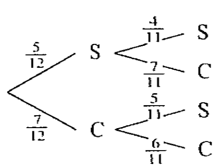
iii P(both hard) $= \frac{4}{11} \times \frac{3}{10}$

$= \frac{6}{55}$

iv P(one of each type) $= \frac{4}{11} \times \frac{7}{10}$

$= \frac{14}{55}$

b i



ii a P(both strawberry) $= \frac{5}{12} \times \frac{4}{11}$

$= \frac{20}{132}$

$= \frac{5}{33}$ (0.152)

b P(second is strawberry) $= \frac{5}{12} \times \frac{4}{11} + \frac{7}{12} \times \frac{5}{11}$

$= \frac{55}{132}$ (0.417)

iii P(selects 4 chocolates) $= \frac{7}{12} \times \frac{6}{11} \times \frac{5}{10} \times \frac{5}{9}$

$= \frac{70}{396}$

$= \frac{35}{198}$ (0.177)

SOLUTIONS TO TOPIC 4 — FUNCTIONS

Short questions

1 a $m = \frac{y_2 - y_1}{x_2 - x_1}$ y-intercept = 1

$\therefore m = \frac{2-1}{2-0}$

$\therefore m = \frac{1}{2}$

and so $C(t) = \frac{1}{2}t + 1$

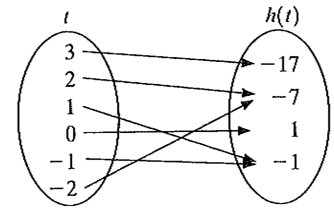
b For $t = 23$, $C(23) = \frac{1}{2}(23) + 1$

$C(23) = \$12.50$

c $\frac{1}{2}t + 1 = 18.31$, $t = 34.62$

$\therefore$ approximate time is 35 minutes.

2 a $h(t) = 1 - 2t^2$



b $D = \{-2, -1, 0, 1, 2, 3\}$

c $R = \{-17, -7, -1, 1\}$

3 a 61% **b** 5.6 years

c At $t = 19$, $P = 100 \times 2^{-\frac{5 \times 19}{28}}$

$\therefore P = 9.52\%$ (3 s.f.)

d Asymptote has equation $P = 0$

4 a $f(2) = 15 - 2(2) = 11$

b $g(-2) = 2^{-2} + 1 = 1\frac{1}{4}$

c $15 - 2x = 2^x + 1$ when $x = 3$ {gdc}

5 a $N(0) = 120 \times (1.04)^0 = 120$

$\therefore$ settlement starts with 120.

b $N(4) = 120 \times (1.04)^4 \approx 140.4$

$\therefore$ 140 people after 4 years.

c $N(t) = 240$ (double)

$\therefore 120 \times (1.04)^t = 240$

$\therefore (1.04)^t = \frac{240}{120} = 2$

$\therefore t = 17.7$ {using a gdc}

$\therefore$ 18 years for the population to double.

6 a $y = ax^2 + bx + c$

If y-int $= 9 \Rightarrow c = 9$

$\therefore y = ax^2 + bx + 9$

b Axis of symmetry is $x = 1$ $\therefore \frac{-b}{2a} = 1$

$\therefore -b = 2a$

$\therefore 2a + b = 0$ ...... (1)

c $(1, 7)$ is a known point.

$\therefore a(1)^2 + b(1) + 9 = 7$

$a + b = -2$ ...... (2)

d Solving equation (1) and (2) simultaneously (gdc)

$a = 2$, $b = -4$

Hence, $a = 2$, $b = -4$ and $c = 9$

7 Graphing $y = x^3 - 3x^2 - x + 3$ on $-2 \leqslant x \leqslant 3$:

a