Calculus Review Packet

1. Consider the function $f(x) = x^3 - 3x^2 - 24x + 30$.

(a) Write down $f(0)$.

(1)

(b) Find $f'(x)$.

(3)

(c) Find the gradient of the graph of $f(x)$ at the point where $x = 1$.

(2)

The graph of $f(x)$ has a local maximum point, M, and a local minimum point, N.

(d) (i) Use $f'(x)$ to find the x-coordinate of M and of N.

(ii) Hence or otherwise write down the coordinates of M and of N.

(5)

(e) Sketch the graph of $f(x)$ for $-5 \leq x \leq 7$ and $-60 \leq y \leq 60$. Mark clearly M and N on your graph.

(4)

Lines L_1 and L_2 are parallel, and they are tangents to the graph of $f(x)$ at points A and B respectively. L_1 has equation $y = 21x + 111$.

(f) (i) Find the x-coordinate of A and of B.

(ii) Find the y-coordinate of B.

(6)
(Total 21 marks)

2. The curve $y = px^2 + qx - 4$ passes through the point $(2, -10)$.

(a) Use the above information to write down an equation in p and q.

(2)

The gradient of the curve $y = px^2 + qx - 4$ at the point $(2, -10)$ is 1.

(b) (i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(ii) Hence, find a second equation in p and q.

(3)

(c) Solve the equations to find the value of p and of q.

(3)
(Total 8 marks)

3. (a) Sketch the graph of $y = 2^x$ for $-2 \le x \le 3$. Indicate clearly where the curve intersects the y-axis.

(3)

(b) Write down the equation of the asymptote of the graph of $y = 2^x$.

(2)

(c) On the same axes sketch the graph of $y = 3 + 2x - x^2$. Indicate clearly where this curve intersects the x and y axes.

(3)

(d) Using your graphic display calculator, solve the equation $3 + 2x - x^2 = 2^x$.

(2)

(e) Write down the maximum value of the function $f(x) = 3 + 2x - x^2$.

(1)

(f) Use Differential Calculus to verify that your answer to (e) is correct.

(5)
(Total 16 marks)

4. Let $f(x) = 2x^2 + x - 6$

(a) Find $f'(x)$.

(3)

(b) Find the value of $f'(-3)$.

(1)

(c) Find the value of x for which $f'(x) = 0$.

(Total 6 marks)

5. Consider the curve $y = x^3 + \frac{3}{2}x^2 - 6x - 2$

(a) (i) Write down the value of y when x is 2.

(ii) Write down the coordinates of the point where the curve intercepts the y-axis.

(3)

(b) Sketch the curve for $-4 \le x \le 3$ and $-10 \le y \le 10$. Indicate clearly the information found in (a).

(4)

(c) Find $\frac{dy}{dx}$.

(3)

(d) Let L_1 be the tangent to the curve at $x = 2$.
Let L_2 be a tangent to the curve, parallel to L_1.

(i) Show that the gradient of L_1 is 12.

(ii) Find the x-coordinate of the point at which L_2 and the curve meet.

(iii) Sketch and label L_1 and L_2 on the diagram drawn in (b).

(8)

(e) It is known that $\frac{dy}{dx} > 0$ for $x < -2$ and $x > b$ where b is positive.

(i) Using your graphic display calculator, or otherwise, find the value of b.

(ii) Describe the behaviour of the curve in the interval $-2 < x < b$.

(iii) Write down the equation of the tangent to the curve at $x = -2$.

(5)
(Total 23 marks)

6. *It is **not** necessary to use graph paper for this question.*

(a) Sketch the curve of the function $f(x) = x^3 - 2x^2 + x - 3$ for values of x from -2 to 4, giving the intercepts with both axes.

(3)

(b) On the same diagram, sketch the line $y = 7 - 2x$ and find the coordinates of the point of intersection of the line with the curve.

(3)

(c) Find the value of the gradient of the curve where $x = 1.7$.

(2)
(Total 8 marks)

8. A function is represented by the equation

$$f(x) = ax^2 + \frac{4}{x} - 3.$$

(a) Find $f'(x)$.

(3)

The function $f(x)$ has a local maximum at the point where $x = -1$.

(b) Find the value of a.

(3)
(Total 6 marks)

15. Consider the function $f(x) = x^3 + 7x^2 - 5x + 4$.

(a) Differentiate $f(x)$ with respect to x.

(3)

(b) Calculate $f'(x)$ when $x = 1$.

(2)

(c) Calculate the values of x when $f'(x) = 0$.

(3)

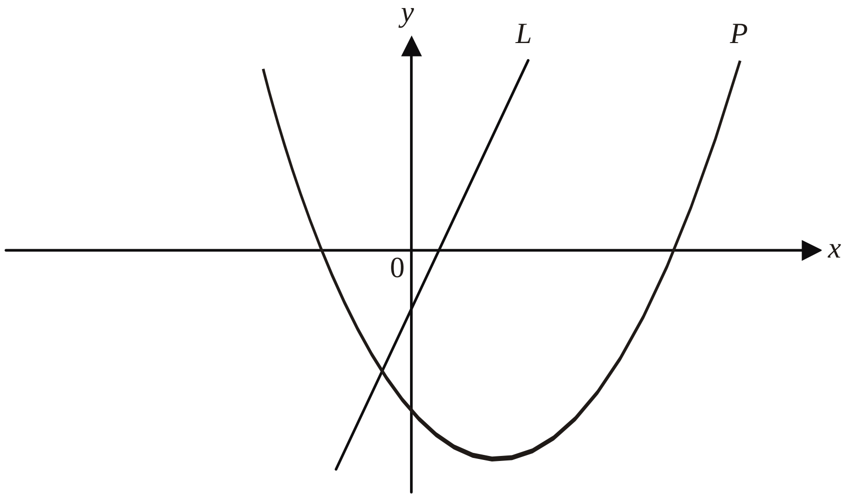
(d) Calculate the coordinates of the local maximum and the local minimum points.

(2)

(e) On graph paper, taking axes $-6 \le x \le 3$ and $0 \le y \le 80$, draw the graph of $f(x)$ indicating clearly the local maximum, local minimum and y-intercept.

(4)
(Total 14 marks)

7. The diagram below shows the graph of a line L passing through (1, 1) and (2, 3) and the graph P of the function $f(x) = x^2 - 3x - 4$



(a) Find the gradient of the line L.

(2)

(b) Differentiate $f(x)$.

(2)

(c) Find the coordinates of the point where the tangent to P is parallel to the line L.

(3)

(d) Find the coordinates of the point where the tangent to P is perpendicular to the line L.

(4)

(e) Find

(i) the gradient of the tangent to P at the point with coordinates $(2, -6)$;

(ii) the equation of the tangent to P at this point.

(3)

(f) State the equation of the axis of symmetry of P.

(1)

(e) Find the coordinates of the vertex of P and state the gradient of the curve at this point.

(3)
(Total 18 marks)

9. A farmer has a rectangular enclosure with a straight hedge running down one side. The area of the enclosure is 162 m^2. He encloses this area using x metres of the hedge on one side as shown on the diagram below.

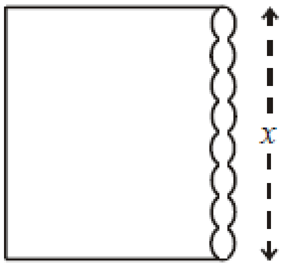


diagram not to scale

(a) If he uses y metres of fencing to complete the enclosure, show that $y = x + \dfrac{324}{x}$.

(3)

The farmer wishes to use the least amount of fencing.

(b) Find $\dfrac{dy}{dx}$.

(3)

(c) Find the value of x which makes y a minimum.

(3)

(d) Calculate this minimum value of y.

(2)

(e) Using $y = x + \dfrac{324}{x}$ find the values of a and b in the following table.

x	6	9	12	18	24	27	36
y	60	45	39	a	37.5	b	45

(2)

(f) Draw an accurate graph of this function using a horizontal scale starting at 0 and taking 2 cm to represent 10 metres, and a vertical scale **starting at 30** with 4 cm to represent 10 metres.

(5)

(g) Write down the values of x for which y increases.

(2)
(Total 20 marks)

10. (a) Expand the expression $2x(x^2 - 1)$.

(b) Hence differentiate $f(x) = 2x(x^2 - 1)$ with respect to x.

(c) Find the gradient of the tangent to the curve $y = f(x)$ at the point where $x = -1$.

(d) If the angle between the x-axis and the tangent in part (c) is θ, write down the value of $\tan\theta$.

(Total 6 marks)

11. Consider the function $f(x) = \dfrac{3}{x^2} + x - 4$.

(a) Calculate the value of $f(x)$ when $x = 1$.

(2)

(b) Differentiate $f(x)$.

(4)

(c) Find $f'(1)$.

(2)

(d) Explain what $f'(1)$ represents.

(2)

(e) Find the equation of the tangent to the curve $f(x)$ at the point where $x = 1$.

(3)

(f) Determine the x-coordinate of the point where the gradient of the curve is zero.

(3)

(Total 16 marks)

12. (a) Differentiate the following function with respect to x:

$$f(x) = 2x - 9 - 25x^{-1}$$

(b) Calculate the x-coordinates of the points on the curve where the gradient of the tangent to the curve is equal to 6.

(Total 6 marks)

13. (a) Differentiate the function $y = x^2 + 3x - 2$.

(b) At a certain point (x, y) on this curve the gradient is 5. Find the co-ordinates of this point.

(Total 6 marks)

14. (a) On the same graph sketch the curves $y = x^2$ and $y = 3 - \frac{1}{x}$ for values of x from 0 to 4 and values of y from 0 to 4. Show your scales on your axes.

(4)

(b) Find the points of intersection of these two curves.

(4)

(c) (i) Find the gradient of the curve $y = 3 - \frac{1}{x}$ in terms of x.

(ii) Find the value of this gradient at the point (1, 2).

(4)

(d) Find the equation of the tangent to the curve $y = 3 - \frac{1}{x}$ at the point (1, 2).

(3)

(Total 15 marks)

17. (a) Write $\frac{3}{x^2}$ in the form $3x^a$ where $a \in \mathbb{Z}$.

(b) Hence differentiate $y = \frac{3}{x^2}$ giving your answer in the form $\frac{b}{x^c}$ where $c \in \mathbb{Z}^+$.

(Total 6 marks)

16. A closed box has a square base of side x and height h.

(a) Write down an expression for the volume, V, of the box.

(1)

(b) Write down an expression for the total surface area, A, of the box.

(1)

The volume of the box is 1000 cm^3

(c) Express h in terms of x.

(2)

(d) Hence show that $A = 4000x^{-1} + 2x^2$.

(2)

(e) Find $\frac{dA}{dx}$.

(2)

(f) Calculate the value of x that gives a minimum surface area.

(4)

(g) Find the surface area for this value of x.

(3)
(Total 15 marks)

20. A function is given as $y = ax^2 + bx + 6$.

(a) Find $\frac{dy}{dx}$.

(2)

(b) If the gradient of this function is 2 when x is 6 write an equation in terms of a and b.

(2)

(c) If the point (3, –15) lies on the graph of the function find a second equation in terms of a and b.

(2)
(Total 6 marks)

18. At the circus a clown is swinging from an elastic rope. A student decides to investigate the motion of the clown. The results can be shown on the graph of the function $f(x) = (0.8^x)(5 \sin 100x)$, where x is the horizontal distance in metres.

(a) Sketch the graph of $f(x)$ for $0 \le x \le 10$ and $-3 \le f(x) \le 5$.

(5)

(b) Find the coordinates of the first local maximum point.

(2)

(c) Find the coordinates of one point where the curve cuts the y-axis.

(1)

Another clown is fired from a cannon. The clown passes through the points given in the table below:

Horizontal distance (x)	Vertical distance (y)
0.00341	0.0102
0.0238	0.0714
0.563	1.69
1.92	5.76
3.40	10.2

(d) Find the correlation coefficient, r, and comment on the value for r.

(3)

(e) Write down the equation of the regression line of y on x.

(2)

(f) Sketch this line on the graph of $f(x)$ in part (a).

(1)

(g) Find the coordinates of one of the points where this line cuts the curve.

(2)
(Total 16 marks)

19. The cost of producing a mathematics textbook is \$15 (US dollars) and it is then sold for \$$x$.

(a) Find an expression for the profit made on each book sold.

(1)

A total of (100 000 – 4000x) books is sold.

(b) Show that the profit made on all the books sold is

$$P = 160\,000x - 4000x^2 - 1500\,000.$$

(3)

(c) (i) Find $\frac{dP}{dx}$.

(2)

(ii) Hence calculate the value of x to make a maximum profit

(2)

(d) Calculate the number of books sold to make this maximum profit.

(2)
(Total 10 marks)

22. Consider the function $g(x) = x^4 + 3x^3 + 2x^2 + x + 4$.

Find

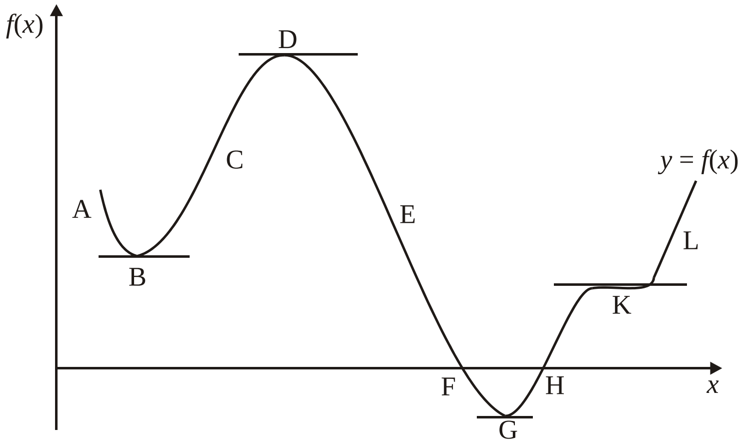
(a) $g'(x)$

(3)

(b) g' (l)

(2)
(Total 5 marks)

21.



Given the graph of $f(x)$ state

(a) the intervals from A to L in which $f(x)$ is increasing.

(1)

(b) the intervals from A to L in which $f(x)$ is decreasing.

(1)

(c) a point that is a maximum value.

(1)

(d) a point that is a minimum value.

(1)

(e) the name given to point K where the gradient is zero.

(1)
(Total 5 marks)

23. A function $g(x) = x^3 + 6x^2 + 12x + 18$

(a) Find $g'(x)$.

(3)

(b) Solve $g'(x) = 0$.

(2)

(c) (i) Calculate the values of $g'(x)$ when

(a) $x = -3$;

(b) $x = 0$.

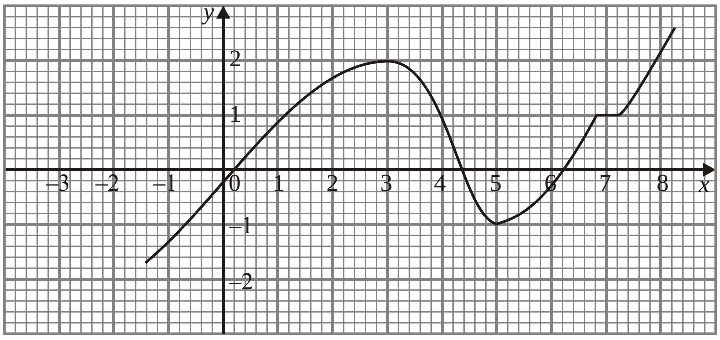
(ii) Hence state whether the function is increasing or decreasing at

(a) $x = -3$;

(b) $x = 0$.

(4)
(Total 9 marks)

24. The diagram shows a part of the curve $y = f(x)$.



(a) For what values of x is $f'(x) = 0$?

(3)

(b) For what range of values of x is $f'(x) < 0$?

(2)
(Total 5 marks)

25. Consider the function $f(x) = 2x^3 - 3x^2 - 12x + 5$.

(a) (i) Find $f'(x)$.

(ii) Find the gradient of the curve $f(x)$ when $x = 3$.

(4)

(b) Find the x-coordinates of the points on the curve where the gradient is equal to –12.

(3)

(c) (i) Calculate the x-coordinates of the local maximum and minimum points.

(ii) Hence find the coordinates of the local minimum.

(6)

(d) For what values of x is the value of $f(x)$ increasing?

(2)
(Total 15 marks)

26. The distance s metres run by an athlete in t minutes is given by the formula

$$s(t) = 250t + 5t^2 - 0.06t^3, \ 0 \le t \le 70.$$

(a) Calculate the distance run after 50 minutes.

(1)

(b) (i) Show that 50 minutes and 1 second may be written as $50\frac{1}{60}$ minutes.

(ii) Calculate the distance run after $50\frac{1}{60}$ minutes.

(iii) Calculate the value of $\dfrac{s\left(50\frac{1}{60}\right) - s(50)}{\frac{1}{60}}$

(5)
(Total 6 marks)

27. The curve $y = f(x)$ has its only local minimum value at $x = a$ and its only local maximum value at $x = b$.

(a) If $a < 0$ and $b > 0$, **sketch** a possible curve of $y = f(x)$ indicating clearly the points $(a, f(a))$ and $(b, f(b))$.

(2)

(b) Given that $0 < h < 1$ and $b - a > 1$, are the following statements about the curve $y = f(x)$ TRUE or FALSE?

(i) $f(a + h) < f(a)$

(ii) $f'(b - h)$ is positive

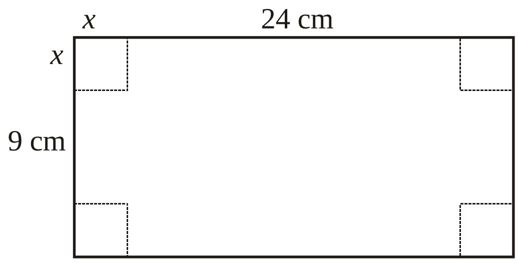
(iii) The tangent to the curve at the point $(a, f(a))$ is parallel to the vertical axis.

(iv) The gradient of the tangent to the curve at the point $(a, f(a))$ is equal to zero.

(v) $f(a - h) < f(a) < f(a + h)$

(5)
(Total 7 marks)

28. A rectangular piece of card measures 24 cm by 9 cm. Equal squares of length x cm are cut from each corner of the card as shown in the diagram below. What is left is then folded to make an **open** box, of length l cm and width w cm.



(a) Write expressions, in terms of x, for

(i) the length, l;

(ii) the width, w.

(2)

(b) Show that the volume (B m^3) of the box is given by $B = 4x^3 - 66x^2 + 216x$.

(1)

(c) Find $\frac{dB}{dx}$.

(1)

(d) (i) Find the value of x which gives the maximum volume of the box.

(ii) Calculate the maximum volume of the box.

(4)
(Total 8 marks)

29. The function g is defined as follows

$$g: x \mapsto px^2 + qx + c, \qquad p, q, c \in \mathbb{R}$$

(a) Find $g'(x)$

(1)

(b) If $g'(x) = 2x + 6$, find the values of p and q.

(2)

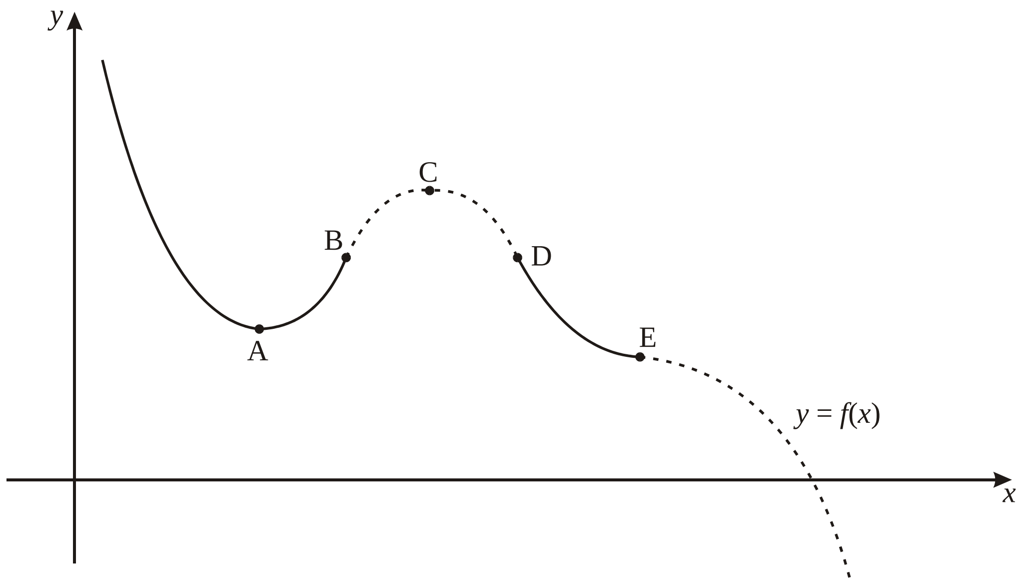
(c) $g(x)$ has a minimum value of –12 at the point A. Find

(i) the x-coordinate of A;

(ii) the value of c.

(4)
(Total 7 marks)

30. The letters A to E are placed at particular points on the curve $y = f(x)$.



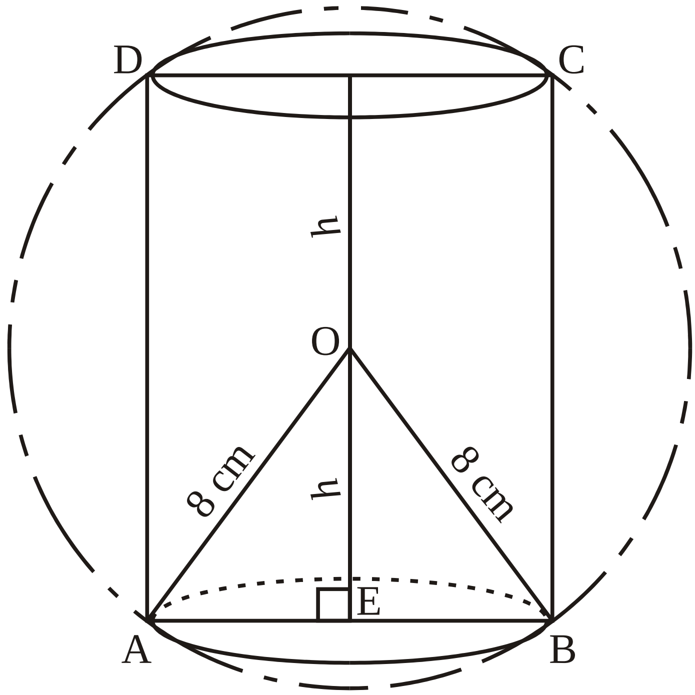
(a) What is the gradient of the curve $y = f(x)$ at the point marked C?

(1)

(b) In passing from point B, through point C, to point D what is happening to $\frac{dy}{dx}$? Is it decreasing or increasing?

(2)
(Total 3 marks)

31. A cylinder is cut from a solid wooden sphere of radius 8 cm as shown in the diagram. The height of the cylinder is $2h$ cm.



(a) Find AE (the radius of the cylinder), in terms of h.

(2)

(b) Show that the volume (V) of the cylinder may be written as

$$V = 2\pi h\,(64 - h^2)\text{ cm}^3.$$

(2)

(c) (i) Determine, correct to three significant figures, the height of the cylinder with the greatest volume that can be produced in this way.

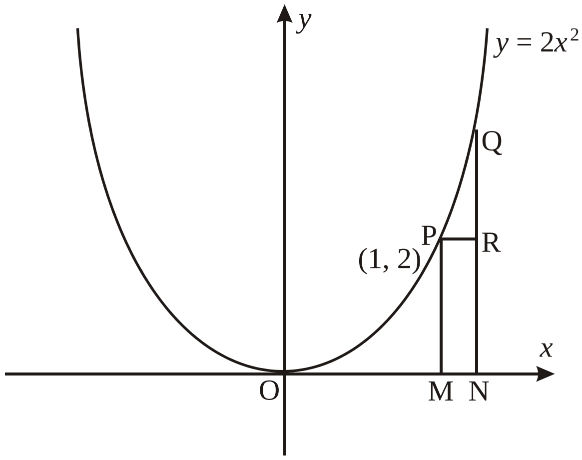
(5)

(ii) Calculate this greatest volume, giving your answer correct to the nearest cm^3.

(3)

(Total 12 marks)

32. The diagram below is a part of the graph of the function $y = 2x^2$. P is the point (1, 2) on the curve and Q is a neighbouring point.



(a) Complete the following table

PR	0.1	0.01	0.001
QN			
QR			
Gradient of PQ			

(4)

(b) Show how you can use your table to

(i) predict the gradient of the curve $y = 2x^2$ at the point (1,2);

(1)

(ii) deduce the gradient of the curve $y = 2x^2 + 3$ at the point (1,5).

(3)
(Total 8 marks)

33. The function $f(x)$ is given by $f(x) = x^3 - 3x^2 + 3x$, for $-1 \le x \le 3$.

(a) Differentiate $f(x)$ with respect to x.

(2)

(b) Complete the table below.

x	–1	0	1	2	3
$f(x)$		0	1	2	9
$f'(x)$	12		0		12

(3)

(c) Use the information in your table to sketch the graph of $f(x)$.

(2)

(d) Write down the gradient of the tangent to the curve at the point (3, 9).

(1)
(Total 8 marks)

34. The perimeter of a rectangle is 24 metres.

(a) The table shows some of the possible dimensions of the rectangle.
Find the values of *a*, *b*, *c*, *d* and *e*.

Length (m)	Width (m)	Area (m^2)
1	11	11
a	10	*b*
3	*c*	27
4	*d*	*e*

(2)

(b) If the length of the rectangle is x m, and the area is A m^2, express A in terms of x only.

(1)

(c) What are the length and width of the rectangle if the area is to be a maximum?

(3)
(Total 6 marks)

(d) Write down the gradient of the tangent to the curve at the point (3, 9).

(1)
(Total 8 marks)