Rational Exponents 1.9

Part 1: Investigating Fractional Conversion Periods

You have seen that the principal of an investment earning compound interest grows exponentially, according to the relationship $A = P(1 + i)^n$. So far, you have calculated an amount at the end of a conversion period. Can the amount be determined for part of a conversion period?

Think, Do, Discuss

1. Marjorie invested \$1000 in a mutual fund that earns interest at 21%/a, compounded annually. Copy and complete the table. Determine the values of the investment at the beginning of the period, at the end of the first year, and at the end of the second year.

Year	0		1		2
Amount (\$)	$1000(1.21)^0 =$		$1000(1.21)^1 =$		$1000(1.21)^2 =$

2. Suppose Marjorie wishes to determine the value of her investment at the end of six months. How could six months be represented as a fraction of a year?
3. In your table, enter the year and the expression for the amount of the investment at the end of six months.
4. Use your calculator to evaluate $(1.21)^{\frac{1}{2}}$. What value do you get? Store this value in your calculator's memory.
5. Multiply 1000 by the number stored in memory. What does this value represent?
6. Multiply the product in step 5 by the value stored in memory. What does this value represent?
7. Multiply the value stored in memory by itself. To what exponent must $(1.21)^{\frac{1}{2}}$ be raised to obtain 1.21? Describe the mathematical relationship between $(1.21)^{\frac{1}{2}}$ and 1.21.
8. Suppose Marjorie wishes to determine the value of her investment at the end of 18 months. How could 18 months be represented as a fraction of a year?

9. In the table you copied from page 73, enter the year and the expression for the amount of the investment at the end of 18 months.

10. Evaluate 1000 × (number stored in memory) × (number stored in memory) × (number stored in memory). What does the product represent?

11. To what exponent must $(1.21)^{\frac{1}{2}}$ be raised to obtain $(1.21)^{\frac{3}{2}}$?

Describe the mathematical relationship between $(1.21)^{\frac{1}{2}}$ and $(1.21)^{\frac{3}{2}}$.

12. Anthony invests $1000 at 9.2727%/a, compounded annually. Determine an expression that represents the amount of the investment at the end of 4 months, 8 months, and 12 months, respectively. Use your calculator to evaluate each expression.

13. Use your calculator to evaluate $(1.092\ 727)^{\frac{1}{3}}$. How many times must this result be multiplied by itself to get 1.092 727?

14. To what exponent must $(1.092\ 727)^{\frac{1}{3}}$ be raised to obtain 1.092 727?

Describe the mathematical relationship between $(1.092\ 727)^{\frac{1}{3}}$ and 1.092 727.

15. Determine the value of Anthony's investment after 32 months.

Part 2: Defining Rational Exponents

You know that an exponent can be an integer. For example,

$$3^2 = 3 \times 3 \qquad 3^{-2} = \frac{1}{3 \times 3} \qquad 3^0 = 1$$
$$= 9 \qquad\qquad = \frac{1}{9}$$

But what does a rational-number exponent mean?

Defining $x^{\frac{1}{2}}$

Apply the exponent law for the power of a power.

$$\begin{aligned}\left(x^{\frac{1}{2}}\right)^2 &= x^{\frac{1}{2} \times 2} \\ &= x^1 \\ &= x\end{aligned}$$

Since $x^{\frac{1}{2}}$ squared equals x, then $x^{\frac{1}{2}}$ must be defined as $\sqrt{x}$.

Therefore,

$$\sqrt{x} \times \sqrt{x} = x \quad \text{and} \quad \begin{aligned} x^{\frac{1}{2}} \times x^{\frac{1}{2}} &= x^{\frac{1}{2} + \frac{1}{2}} \\ &= x \end{aligned}$$

Therefore, $x^{\frac{1}{2}} = \sqrt{x}$. $\sqrt{x}$ is read "the square root of x."

Defining $x^{\frac{1}{3}}$

Again, apply the exponent law for the power of a power.

$$\begin{aligned}\left(x^{\frac{1}{3}}\right)^3 &= x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}\\ &= x^1\\ &= x\end{aligned}$$

Since $x^{\frac{1}{3}}$ cubed equals x, then $x^{\frac{1}{3}}$ must be defined as $\sqrt[3]{x}$.

Therefore, $\sqrt[3]{x} \times \sqrt[3]{x} \times \sqrt[3]{x}$ and

$$\begin{aligned}x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}} &= x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}}\\ &= x\end{aligned}$$

Therefore, $x^{\frac{1}{3}} = \sqrt[3]{x}$. $\sqrt[3]{x}$ is read "the cube root of x."

Defining $x^{\frac{1}{n}}$

By a similar argument, if n is an integer and $n \neq 0$, then

$x^{\frac{1}{n}} = \sqrt[n]{x}$. $\sqrt[n]{x}$ is read "the nth root of x."

Example 1

Evaluate each power.

(a) $49^{\frac{1}{2}}$ (b) $(-64)^{\frac{1}{3}}$ (c) $8^{-\frac{1}{3}}$ (d) $\left(\frac{1}{36}\right)^{\frac{1}{2}}$

Solution

(a) $$\begin{aligned}49^{\frac{1}{2}} &= \sqrt{49}\\ &= 7\end{aligned}$$

(b) $$\begin{aligned}(-64)^{\frac{1}{3}} &= \sqrt[3]{-64}\\ &= -4\end{aligned}$$

(c) $$\begin{aligned}8^{-\frac{1}{3}} &= \frac{1}{8^{\frac{1}{3}}}\\ &= \frac{1}{\sqrt[3]{8}}\\ &= \frac{1}{2}\end{aligned}$$

(d) $$\begin{aligned}\left(\frac{1}{36}\right)^{\frac{1}{2}} &= \sqrt{\frac{1}{36}}\\ &= \frac{1}{6}\end{aligned}$$

Defining $x^{\frac{2}{3}}$

Use the exponent law for power of a power to express $x^{\frac{2}{3}}$ in two ways.

$$\begin{aligned}x^{\frac{2}{3}} &= x^{\frac{1}{3}\times 2}\\ &= (\sqrt[3]{x})^2\end{aligned}$$

or

$$\begin{aligned}x^{\frac{2}{3}} &= x^{2\times\frac{1}{3}}\\ &= \sqrt[3]{x^2}\end{aligned}$$

Defining $x^{\frac{m}{n}}$

If m and n are integers and $n \neq 0$, then $x^{\frac{m}{n}} = (\sqrt[n]{x})^m = \sqrt[n]{x^m}$.

Example 2

Evaluate.

(a) $8^{\frac{2}{3}}$ (b) $-25^{\frac{5}{2}}$ (c) $81^{-\frac{3}{4}}$

Solution

(a)
$$8^{\frac{2}{3}} = (\sqrt[3]{8})^2 = 2^2 = 4$$

(b)
$$-25^{\frac{5}{2}} = -(\sqrt{25})^5 = -(5)^5 = -3125$$

(c)
$$81^{-\frac{3}{4}} = \frac{1}{81^{\frac{3}{4}}} = \frac{1}{(\sqrt[4]{81})^3} = \frac{1}{(3)^3} = \frac{1}{27}$$

Consolidate Your Understanding

1. What exponent can you use to represent the nth root of a number?
2. Use the exponent law for power of a power to express $27^{\frac{2}{3}}$ in two ways.

Focus 1.9

Key Ideas

- The rational exponent of $\frac{1}{n}$ indicates the nth root of the base. If $n > 1$ and $n \in N$, then $x^{\frac{1}{n}} = \sqrt[n]{x}$.
- If the numerator of a rational exponent is *not* 1 and if m and n are both positive integers, then
  - ♦ $x^{\frac{m}{n}} = \sqrt[n]{x^m}$ $x^{\frac{m}{n}}$ means the nth root of the mth power of x.
  - ♦ $x^{\frac{m}{n}} = (\sqrt[n]{x})^m$ $x^{\frac{m}{n}}$ means the mth power of the nth root of x.

  Note that $x^{\frac{m}{n}} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$.

- Evaluate negative rational exponents in the same way as you would evaluate negative integer exponents. If m and n are both positive integers, then

$$x^{-\frac{m}{n}} = \frac{1}{x^{\frac{m}{n}}}$$
$$= \frac{1}{\sqrt[n]{x^m}} \text{ or } \frac{1}{(\sqrt[n]{x})^m}$$

- These symbols $\sqrt{\ }$, $\sqrt[3]{\ }$, $\sqrt[4]{\ }$, … are called radical signs. $\sqrt{x}$ and $\sqrt[4]{81}$ are examples of **radicals**.

Example 3

Evaluate each power.

(a) $16^{\frac{3}{4}}$

(b) $27^{-\frac{2}{3}}$

Solution

(a) $16^{\frac{3}{4}} = (\sqrt[4]{16})^3$
$= 2^3$
$= 8$

For most scientific calculators, enter
[1] [6] [y^x] [(] [3] [÷] [4] [)] [=].

(b) $27^{-\frac{2}{3}} = (\sqrt[3]{27})^{-2}$
$= 3^{-2}$
$= \frac{1}{3^2}$
$= \frac{1}{9}$

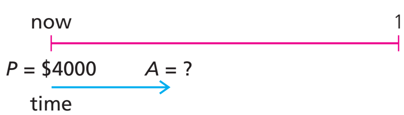
Using the TI-83 Plus calculator, enter
[2] [7] [^] [(-)] [(] [2] [÷] [3] [)] [ENTER]
[MATH] [1] [ENTER].

Example 4

Suppose \$4000 is invested at 5%/a, compounded annually. Determine the amount after 123 days.

Solution

Draw a time line to organize the information.



$A = P(1 + i)^n$, where $P = \$4000$, $i = 5\%$ or 0.05, and $n = \frac{123}{365}$.

$$A = 4000(1.05)^{\frac{123}{365}}$$
$$= 4000(1.016\ 577\ 524)$$
$$= 4066.31$$

The amount after 123 days is \$4066.31.

Example 5

Determine the present value of an investment that will be worth \$8000 in 6 years 8 months. The interest rate is 6%/a, compounded quarterly.

Solution

Draw a time line to organize the information.

$P = A(1 + i)^{-n}$, where $A = \$8000$, $i = \frac{6\%}{4} = 1.5\%$ or 0.015, and $n = \frac{80}{3}$

$$
\begin{aligned}
P &= 8000(1.015)^{-\frac{80}{3}} \\
&= 8000(0.672\,314\,080\,1) \\
&= 5378.51
\end{aligned}
$$

The present value of the investment is \$5378.51.

Example 6

Barb put four new 1.5-V batteries in her portable radio. She went to bed listening to her favourite FM station and fell asleep with the radio on. The radio uses 2% of the charge each hour. If she turned the portable radio off 7.5 h after she had turned it on, determine the remaining charge in the batteries.

Solution

The radio uses 2% of the charge each hour, which means that 98% remains. Since the batteries are new, there are 6 V of charge initially.

Time (h)	0	1	2	3	4
Voltage (V)	6	$6 \times 0.98 = 5.88$	$5.88 \times 0.98 = 5.7624$	$5.7624 \times 0.9 \doteq 5.6472$	$5.6472 \times 0.98 \doteq 5.5343$

The remaining charge after each hour is a geometric sequence, with $a = 6$ and $r = 0.98$. The general term is $t_n = 6(0.98)^{n-1}$.
Evaluate t_n for $n = 8.5$ or $\frac{17}{2}$, since 7.5 h corresponds to $t_{8.5}$.

$$
\begin{aligned}
\text{remaining charge} &= 6(0.98)^{\frac{15}{2}} \\
&= 6(0.859\,400\,432\,1) \\
&\doteq 5.156
\end{aligned}
$$

When Barb turns off the radio, there are 5.156 V remaining in the batteries.

Practise, Apply, Solve 1.9

Answer questions 1 to 4 without the aid of a calculator.

A

1. Evaluate each expression.

(a) 4^3 (b) $(-6)^2$ (c) -3^2 (d) $(-9)^3$ (e) 15^0

(f) $\left(\frac{1}{2}\right)^3$ (g) $\left(\frac{2}{3}\right)^4$ (h) $\left(-\frac{2}{3}\right)^3$ (i) $\left(\frac{3}{4}\right)^{-2}$ (j) $\left(-\frac{1}{4}\right)^{-3}$

2. Evaluate each expression.

(a) $\sqrt{121}$ (b) $\sqrt[3]{8}$ (c) $\sqrt[4]{16}$ (d) $\sqrt[3]{125}$ (e) $\sqrt[5]{1024}$

3. Evaluate each expression.

(a) $4^{\frac{1}{2}}$ (b) $(-8)^{\frac{1}{3}}$ (c) $-81^{\frac{1}{4}}$ (d) $27^{\frac{1}{3}}$ (e) $125^{\frac{1}{3}}$

(f) $\left(\frac{1}{16}\right)^{\frac{1}{2}}$ (g) $(216)^{-\frac{1}{3}}$ (h) $(125)^{-\frac{1}{3}}$ (i) $\left(\frac{4}{9}\right)^{\frac{1}{2}}$ (j) $(16)^{-\frac{1}{4}}$

4. Evaluate each expression.

(a) $16^{\frac{3}{2}}$ (b) $8^{\frac{2}{3}}$ (c) $-16^{\frac{3}{4}}$ (d) $27^{\frac{4}{3}}$ (e) $81^{\frac{5}{4}}$

(f) $\left(\frac{1}{8}\right)^{\frac{2}{3}}$ (g) $(27)^{-\frac{2}{3}}$ (h) $(125)^{-\frac{4}{3}}$ (i) $\left(\frac{4}{25}\right)^{\frac{3}{2}}$ (j) $(32)^{-\frac{3}{5}}$

5. Evaluate to two decimal places. Use a calculator.

(a) $11^{\frac{1}{2}}$ (b) $(-24)^{\frac{1}{3}}$ (c) $-100^{\frac{1}{4}}$ (d) $50^{\frac{1}{3}}$ (e) $165^{\frac{1}{5}}$

(f) $(0.58)^{\frac{1}{2}}$ (g) $(1.05)^{-\frac{1}{3}}$ (h) $(200)^{-\frac{1}{3}}$ (i) $(12.65)^{\frac{1}{6}}$ (j) $(17.6)^{-\frac{1}{4}}$

6. Evaluate to two decimal places. Use a calculator.

(a) $25^{\frac{3}{2}}$ (b) $55^{\frac{2}{3}}$ (c) $1.25^{\frac{3}{4}}$ (d) $12.8^{\frac{2}{5}}$ (e) $167^{\frac{3}{4}}$

(f) $\left(\frac{5}{6}\right)^{\frac{1}{3}}$ (g) $(1034)^{-\frac{2}{3}}$ (h) $(44)^{-\frac{4}{3}}$ (i) $\left(\frac{16}{25}\right)^{\frac{3}{2}}$ (j) $(85)^{-\frac{2}{5}}$

B

7. Determine the amount in each situation.

(a) \$6000 borrowed for 4 months at 3%/a, compounded annually

(b) \$2500 invested for 1 year and 3 months at 4%/a, compounded semiannually

(c) \$12 300 borrowed for 22 months at 4.4%/a, compounded quarterly

(d) \$13 560 invested for 100 days at 6.5%/a, compounded annually

(e) \$3750 financed for 13 months at 3.65%/a, compounded semiannually

(f) \$6320 invested for 6 days at 5.2%/a, compounded weekly

8. Determine the present value

(a) of an investment that will be worth \$3000 in 300 days. The interest rate is 5%/a, compounded annually.

(b) of an investment that will be worth \$10 500 in 4 months. The interest rate is 6%/a, compounded semiannually.

(c) of a loan of \$7400 that will be due in 37 months. The interest rate is 6%/a, compounded quarterly.

(d) of a loan of \$100 000 that will be due in 2 years 5 months. The interest rate is 6.5%/a, compounded semiannually.

(e) of an investment that will be worth \$150 in 8 months. The interest rate is 8.2%/a, compounded annually.

(f) of an investment that will be worth \$3400 in 15 months. The interest rate is 11%/a, compounded semiannually.

9. A car depreciates in value according to the model $V = 25\ 000(0.85)^n$, where V is the value of the car at the end of the nth year.

(a) What was the purchase price of the car?

(b) At what annual rate is the value of the car depreciating?

(c) What is the value of the car at the end of 3 years?

(d) What is the value of the car at the end of 4 years 4 months?

10. A rare stamp, bought for \$500, increases in value by 6% each year.

(a) Determine an algebraic expression to model the stamp's value over time.

(b) Determine the stamp's value five years after it was bought.

(c) Determine the stamp's value 300 days after it was bought.

11. Carbon-16 has a half-life of 20 min.

(a) A sample of carbon-16 has an initial mass of 700 mg. Determine an algebraic expression that models the sample's mass over time.

(b) Determine the sample's mass after 3 h.

(c) What is the sample's mass after 5 min?

12. **Knowledge and Understanding:** To find the depreciation rate, r, of an item, use the formula for declining balances, $r = 1 - \left(\frac{S}{C}\right)^{\frac{1}{n}}$, where S is the salvage value, in dollars, C is the original cost, in dollars, and n is the useful life of the item, in years. Determine the depreciation rate for a truck that was purchased for \$125 000. The salvage value after three years is \$85 000.

13. Many soft drinks contain about 40 mg of caffeine in one 355-mL can. Every 5 h, the mass of caffeine in an adult's bloodstream reduces by half.

(a) Determine an algebraic expression that models the mass of caffeine in an adult's bloodstream over time.

(b) Determine the mass of caffeine in an adult's bloodstream 15 h after he or she drinks a can of cola.

(c) What is the mass of caffeine in an adult's bloodstream 2 h after he or she drinks a can of cola?

14. Simplify.

(a) $(25x^2)^{\frac{1}{2}}$ (b) $(-27x^3)^{\frac{1}{3}}$ (c) $(16x^4)^{\frac{1}{4}}$ (d) $(-16x^4)^{-\frac{1}{2}}$

(e) $(243x^5)^{\frac{1}{5}}$ (f) $[(x+2)^4]^{\frac{1}{4}}$ (g) $(-64x^8)^{\frac{1}{4}}$ (h) $[(2x+5)^7]^{\frac{1}{7}}$

15. **Communication:** Create a flow chart that gives the correct sequence of keystrokes to evaluate $(-125)^{-\frac{5}{3}}$ on a scientific calculator.

16. Simplify.

(a) $(25x^4)^{\frac{3}{2}}$ (b) $(27a^3b^3)^{\frac{2}{3}}$ (c) $\left(\frac{8x^3}{125}\right)^{\frac{2}{3}}$ (d) $\left(\frac{27}{64y^3}\right)^{\frac{2}{3}}$

17. Simplify.

(a) $(16x^2)^{-\frac{3}{2}}$ (b) $(49c^6)^{-\frac{3}{2}}$ (c) $\left(\frac{-8x^3}{216}\right)^{-\frac{1}{3}}$ (d) $\left(\frac{16}{81y^8}\right)^{-\frac{3}{4}}$

18. **Application:** Four years two months ago, Sami invested a sum of money at 5%/a, compounded semiannually. Today there is \$921.35 in the account. How much did Sami invest?

19. A new 1.5-V battery is put in a watch. The battery loses 1% of its charge each day, and the watch needs 0.001 V to run.

(a) Determine whether the watch will still run after 500 days.

(b) Determine the voltage in the battery after 3 h.

20. The length, L, of the longest board that can be carried horizontally around the right-angle corner of two intersecting hallways is $L = \left(a^{\frac{2}{3}} + b^{\frac{2}{3}}\right)^{\frac{3}{2}}$, where a and b represent the width, in centimetres, of the hallways. What is the longest piece of plywood a carpenter can carry horizontally around the corner of two intersecting hallways if one hallway is 150 cm wide and the other is 200 cm wide? Round to the nearest hundredth.

21. **Check Your Understanding**

(a) Evaluate $(32)^{-\frac{3}{5}}$ using two different methods. Do not use a calculator.

(b) Use a scientific calculator to verify your answers.

C

22. **Thinking, Inquiry, Problem Solving:** Leon invested a sum of money ten years ago. Today the investment has grown to \$7146.03. He invested the original amount for 4 years 6 months at 8%/a, compounded quarterly. Then he reinvested this amount at 6%/a, compounded monthly.

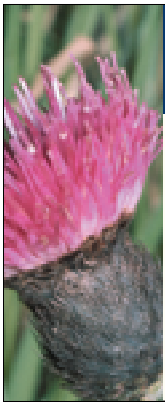
(a) How much did Leon originally invest?

(b) Suppose instead that he had invested the principal for ten years in an account that paid 10%/a, compounded semiannually. What must be the interest rate to generate the same amount, \$7146.03?

23. The time, t, in minutes, of a communications satellite's orbit of the Earth is $t = (1.66 \times 10^{-4})(6370 + h)^{1.5}$, where h is the height, in kilometres, of the satellite above the Earth.

(a) Determine the time for a satellite that is 450 km above the Earth to orbit the Earth once.

(b) If television satellites must orbit the Earth at the same rate that the Earth rotates, then determine how high above the Earth's surface a television satellite must be for this to happen.



The Chapter Problem—Controlling Non-Native Plant Populations

In this section, you have used rational exponents. Apply what you have learned to answer these questions about the Chapter Problem on page 12.

CP16. Use your model to predict the number of plants present after 5 years.

CP17. Use your model to predict the number of seeds present after 2 years 3 months.

CP18. What factors make question CP17 unreasonable for this situation?

Did You Know?

Go is a game played on a board with 19 horizontal and vertical lines. Players take turns trying to capture territory by placing pebbles where the lines intersect. The winner of the game has the most occupied territory and pieces.

Computers can now beat the greatest chess players, but they cannot yet beat the best Go players. This is because there are too many moves for even the fastest computers to calculate.