Lesson 56B – Second Derivatives, Inflection Points and Concavity

SL1 Math - Santowski

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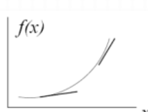
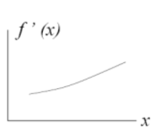
Lesson Objectives

- 1. Calculate second derivatives of functions
- 2. Define concavity and inflection point
- 3. Test for concavity in a function using the second derivative
- 4. Test extrema/critical points using the second derivative

(A) New Term – Concave Up

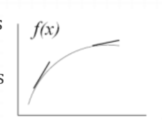
- Concavity is best "defined" with graphs
- If f''(x) > 0 then f'(x) is increasing. This means that the slope of the original function is getting steeper (from left to right). The function curves upwards: we say that it is ***concave up***.
- The curve "opens upwards" or "curves up".

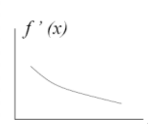




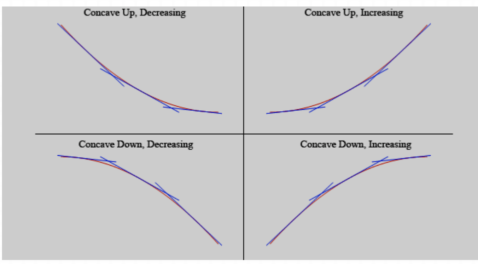
(B) New Term – Concave down

- Concavity is best "defined" with graphs
- If f''(x) < 0 then f'(x) is decreasing. This means that the slope of the original function is getting shallower (from left to right). The function curves downwards: we say that it is ***concave downward***.
- The curve "opens downwards" or "curves down".





(B) New Term – Concavity

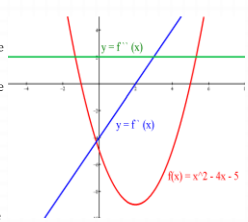


(C) Example #1

- For the function $f(x) = x^2 - 4x - 5$, determine the equations of the first and second derivatives of f(x)
- Graph the function and its first and second derivatives on the same grid

(C) Example #1

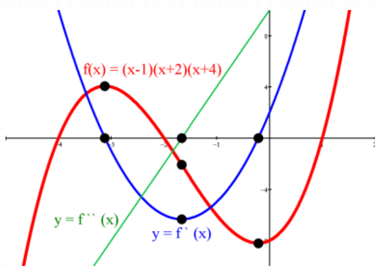
- Points to note:
- (1) the fcn has a minimum at x=2 and the derivative has an x-intercept at x=2
- (2) the fcn decreases on (-∞,2) and the derivative has negative values on (-∞,2)
- (3) the fcn increases on (2,+∞) and the derivative has positive values on (2,+∞)
- (4) the fcn changes from decrease to increase at the min while the derivative values change from negative to positive
- (5) the function is concave up and the derivative fcn is an increasing fcn
- (6) the second derivative graph is positive on the entire domain



(C) Example #2

- For the function $f(x) = x^3 + 5x^2 + 2x - 8$, determine the equations of the first and second derivatives of f(x)
- Graph the function and its first and second derivatives on the same grid

(C) Example #2



(C) Example #2 - FD

- f(x) has a max. at x = -3.1 and f`(x) has an x-intercept at x = -3.1
- f(x) has a min. at x = -0.2 and f`(x) has a root at –0.2
- f(x) increases on (-∞, -3.1) & (-0.2, ∞) and on the same intervals, f`(x) has positive values
- f(x) decreases on (-3.1, -0.2) and on the same interval, f`(x) has negative values
- At the max (x = -3.1), the fcn changes from being an increasing fcn to a decreasing fcn ➔ the derivative changes from positive values to negative values
- At a the min (x = -0.2), the fcn changes from decreasing to increasing ➔ the derivative changes from negative to positive

(C) Example #2

- f(x) is concave down on (-∞, -1.67) while f`(x) decreases on (-∞, -1.67) and the 2nd derivative is negative on (-∞, -1.67)
- f(x) is concave up on (-1.67, ∞) while f`(x) increases on (-1.67, ∞) and the 2nd derivative is positive on (-1.67, ∞)
- The concavity of f(x) changes from CD to CU at x = -1.67, while the derivative has a min. at x = -1.67

- Since the function is concave down where x = -3.1, the function has a maximum point at x = -3.1
- Since the function is concave up where x = -0.2, the function has a minimum point at x = -0.2

(D) Second Derivative – A Summary

- If f``(x) > 0 at x = ***p***, then f(x) is concave up at x = ***p***
- If f``(x) > 0 on the interval ***(a,b)***, then f(x) is concave up on ***(a,b)***

- If f``(x) < 0 at x = ***p***, then f(x) is concave down at x = ***p***
- If f``(x) < 0 on the interval ***(a,b)***, then f(x) is concave down on ***(a,b)***

- If f``(x) = 0, then f(x) is neither concave nor concave down, but MAY HAVE an ***inflection point*** ➔ a point at which the concavity is then changing directions

- So to test whether concavity changes, test the sign of the second derivative before and after the point where f''(x) = 0

(E) Second Derivative Test

- the "Second Derivative Test" allows us to test for maximum and minimum values
- The second derivative also gives information about the "extreme points" or "critical points" or max/mins on the original function:
  - If f`(x) = 0 and f``(x) > 0, then the critical point is a minimum point (picture $y = x^2$ at x = 0)
  - If f`(x) = 0 and f``(x) < 0, then the critical point is a maximum point (picture $y = -x^2$ at x = 0)

(F) Examples - Algebraically

- Find where the curve $y = x^3 - 3x^2 - 9x - 5$ is concave up and concave down. Find and classify all extreme points. Then use this info to sketch the curve.

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- $f(x) = x^3 - 3x^2 - 9x - 5$
- $f`(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x - 3)(x + 1)$
- So f(x) has critical points (or local/global extrema) at x = -1 and x = 3

- $f``(x) = 6x - 6 = 6(x - 1)$
- So at x = 1, f``(x) = 0 and we have a change of concavity
- Then f ``(-1) = -12 ➔ the curve is concave down, so x = -1 must represent a maximum point
- Also f`(3) = +12 ➔ the curve is concave up, so x = 3 must represent a minimum point
- Then f(3) = -33, f(-1) = 0 as the ordered pairs for the function whose graph is shown on the next slide:

(G) In Class Examples

- ex 1. Find and classify all local extrema of $f(x) = 3x^5 - 25x^3 + 60x$. Sketch the curve
- ex 2. Find and classify all local extrema of $f(x) = 3x^4 - 16x^3 + 18x^2 + 2$. Sketch the curve
- ex 3. Find where the curve $y = x^3 - 3x^2$ is concave up and concave down. Then use this info to sketch the curve

(G) Example

- Ex 4. Find where the curve $y = 4x^3 - 3x^2 + 1$ is concave up and concave down and determine the co-ordinates of the inflection point(s). Then use this info to sketch the curve
- Ex 5. Determine the intervals of concavity and inflection points of $f(x) = 3x^5 – 5x^3 + 3$. For this question, you will solve graphically and then verify algebraically

(H) Internet Links

- We will work on the following problems in class: Graphing Using First and Second Derivatives from UC Davis
- Visual Calculus - Graphs and Derivatives from UTK
- Calculus I (Math 2413) - Applications of Derivatives - The Shape of a Graph, Part II Using the Second Derivative - from Paul Dawkins
- http://www.geocities.com/CapeCanaveral/Launchpad/2426/page203.html

(I) Homework

- IB Math, photocopy from Stewart, 1997, Calculus – Concepts and Contexts, p292, Q1-26
- MCB4U – Nelson text, p329, Q1–13,15