Lesson 47 – Trigonometric Functions & Periodic Phenomenon

IB Math SL1 - Santowski

Lesson Objectives

- 1. Relate the features of sinusoidal curves to modeling periodic phenomenon
- 2. Transformations of sinusoidal functions and their features

(A) Key Terms

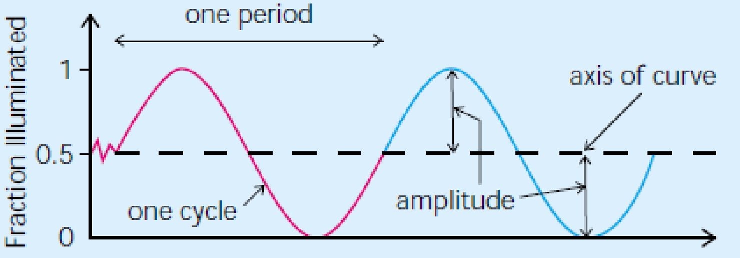
- Define the following key terms that relate to trigonometric functions:

- (a) period
- (b) amplitude
- (c) axis of the curve (or equilibrium axis)

(A) Key Terms

Key Ideas

- Repeating data forms a periodic function.
- A periodic function has a self-repeating graph.
- The **cycle** of a graph is the smallest complete repeating pattern of the graph.



- The length of one cycle is called the **period**.
- The horizontal line that is halfway between the maximum and minimum values of a periodic curve is called the **axis of the curve**.
- The equation of the axis of the curve is

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$

- The magnitude of the vertical distance from the axis of the curve to either the maximum or minimum value is called the **amplitude** of the function. The amplitude, a, is calculated as

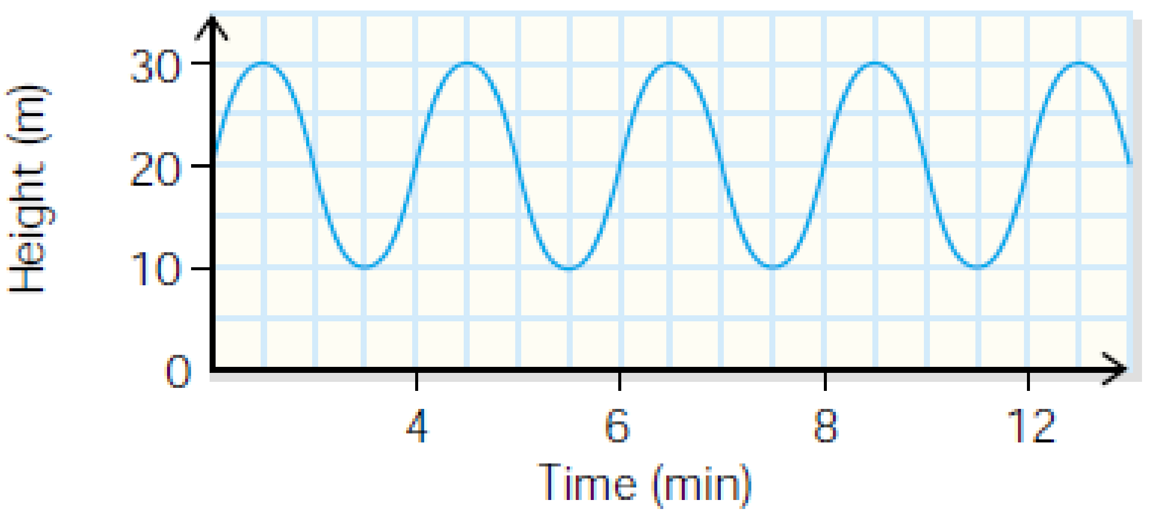
$$a = \frac{\text{maximum value} - \text{minimum value}}{2}$$

The General Sinusoidal Equation

- In the equation f(x) = ***a***sin(***k***(x+***c***)) + ***d***, explain what:
- a represents?
- k represents?
- c represents?
- d represents?

(A) Fast Five

- The graph shows John's height above the ground as a function of time as he rides a Ferris wheel.
  - **(a)** State the maximum and minimum height of the ride.
  - **(b)** How long does the Ferris wheel take to make one complete revolution?
  - **(c)** What is the amplitude of the curve? How does this relate to the Ferris wheel?
  - **(d)** Determine the equation of the axis of the curve.



(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

Application: A buoy rises and falls as it rides the waves. The equation $h(t) = \cos \frac{\pi}{5} t$ models the displacement of the buoy in metres at t seconds.

(a) Graph the displacement from 0 to 20 s, in 2.5-s intervals.

(b) Determine the period of the function from the graph. Determine the period algebraically from the equation.

(c) What is the displacement at 35 s?

(d) At what time, to the nearest second, does the displacement first reach -0.8 m?

(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

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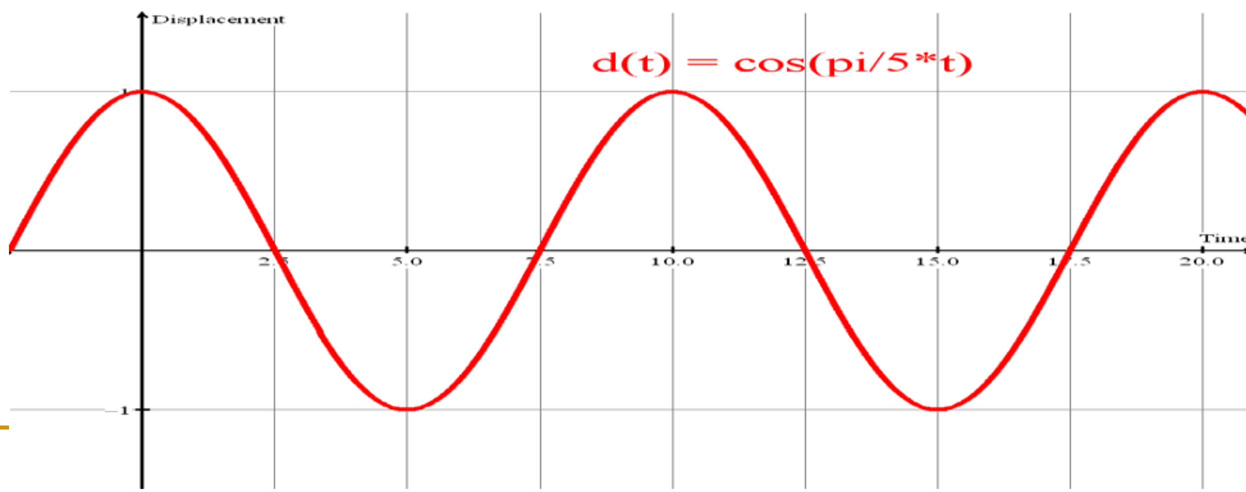


(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

The height, h, of a basket on a water wheel at time t is given by $h(t) = \sin(6t)^\circ$, where t is in seconds and h is in metres.

(a) How high is the basket at 14 s?

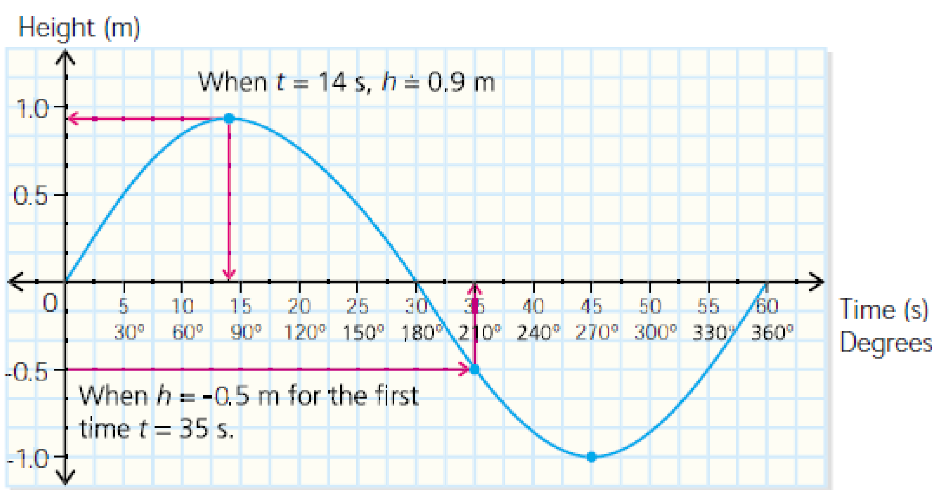
(b) When will the basket first be 0.5 m under water?

(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

(a) The values of h and t can be interpolated from a graph. Prepare a table. In this case, 5-s intervals were used, although the interval size could be different.

t (seconds)	0	5	10	15	20	25	30	35	40	45	50	55	60
$(6t)°$	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$h(t) = \sin (6t)°$ (metres)	0	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0

Interpolating from the graph gives a value of about 0.9 m, as shown.



(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

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(b) When will the basket first be 0.5 m under water?

The question could also be answered by substituting into the equation.

Then,

$$\begin{aligned} h(14) &= \sin(6 \times 14)^\circ \\ &= \sin 84^\circ \\ &\doteq 0.995 \end{aligned}$$

Substitution gives a more accurate answer in this case. At 14 s, the height is almost 1 m.

A height of 0.5 m under water corresponds to a height of -0.5 m in the model. Therefore,

$$\begin{aligned} h(t) &= \sin(6t)^\circ \\ -0.5 &= \sin(6t)^\circ \end{aligned}$$

Interpolation shows that $\sin 210^\circ = -0.5$.

The basket will be 0.5 m under water for the first time at 35 s.

(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

- A spring bounces up and down according to the model $d(t) = 0.5 \cos 2t$, where d is the displacement in centimetres from the rest position and t is the time in seconds. The model does not consider the effects of gravity.

  - **(a)** Make a table for $0 \leq t \leq 9$, using 0.5-s intervals.
  - **(b)** Draw the graph.
  - **(c)** Explain why the function models periodic behaviour.
  - **(d)** What is the relationship between the amplitude of the function and the displacement of the spring from its rest position?
  - **(e)** What is the period and what does it represent in the context of this question?
  - **(f)** What is the amplitude and what does it represent in the context of this question?

(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

Look up at the moon on a clear night. Sometimes the moon is full and the night sky is bright. At other times, there is a new moon with no visible light and the sky is dark. The moon is said to wax from dark to bright and wane back to dark.

Fraction of the Moon Visible at Midnight
Days 1 to 66 of the Year 2000

Day of the Year	1	2	3	4	5	6	10	15	20	21
Fraction of Moon Visible	0.25	0.18	0.11	0.06	0.02	0.00	0.11	0.57	0.99	1.00

Day of the Year	25	30	35	40	45	50	55	60	65	66
Fraction of Moon Visible	0.80	0.32	0.02	0.14	0.64	1.00	0.77	0.31	0.01	0.00

(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

- **1.** Draw and label a scatter plot of the data. Then draw the curve of best fit.

- **2. (a)** Starting with day 1, how many days does it take for the shortest complete pattern of the graph to repeat?
- **(b)** Starting with day 6, how many days does the graph take to repeat?
- **(c)** On what other day could the graph begin and still repeat?

- **3. (a)** Extend the pattern of the graph to include the 95th day of the new millennium. Was the phase of the moon closer to a full moon or a new moon? Explain.

- **(b)** Extend the graph to predict the fraction of the moon that was visible on the summer solstice, June 21. Was the moon waxing or waning? Explain.

(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

- You found that this data represents a periodic phenomenon with the following properties:

- The period is about 29.5 days.
- The “full” moon is fully visible when the maximum value is 1.0.
- The “new” moon is not visible when the minimum value is 0.
- The axis of the curve is the horizontal line *y* 0.5.
- The amplitude of the curve is 0.5.

- You know that a sinusoidal model of this data is:
- f(x) = ***a***sin(***k***(x+***c***)) + ***d***

(E) Combining Transformations

- We continue our investigation by graphing some other functions in which we have combined our transformations
- (i) Graph and analyze $y = 2 \sin 3(x - 60^\circ) + 1$ ➔ identify transformations and state how the key features have changed
- (ii) Graph and analyze $y = 2 \cos [2(x - \pi/4)] – 3$ ➔ identify transformations and state how the key features have changed
- (iii) Graph and analyze $y = \tan(\frac{1}{2} x + \pi/4) – 3$ ➔ identify transformations and state how the key features have changed

(B) Writing Sinusoidal Equations

- ex 1. Given the equation $y = 2\sin 3(x - 60^\circ) + 1$, determine the new amplitude, period, phase shift and equation of the axis of the curve.

- Amplitude is obviously 2
- Period is $2\pi/3$ or $360^\circ/3 = 120^\circ$
- The equation of the equilibrium axis is $y = 1$
- The phase shift is 60° to the right

(B) Writing Sinusoidal Equations

- ex 2. Given a cosine curve with an amplitude of 2, a period of 180°, an equilibrium axis at y = -3 and a phase shift of 45° right, write its equation.

- So the equation is $y = 2\cos[2(x - 45°)] - 3$

- Recall that the k value is determined by the equation period = $2\pi/k$ or $k = 2\pi/\text{period}$
- If working in degrees, the equation is modified to period = $360°/k$ or $k = 360°/\text{period}$

(C) Writing Sinusoidal Equations from Word Problems

- Now we shift to word problems wherein we must carry out the same skills in order to generate an equation for the sinusoidal curve that best models the situation being presented.

- ex 5. A small windmill has its center 6 m above the ground and the blades are 2 m in length. In a steady wind, one blade makes a rotation in 12 sec. Use the point *P* as a reference point on a blade that started at the highest point above the ground.

- (a) Determine an equation of the function that relates the height of a tip of a blade, *h* in meters, above the ground at a time *t*.
- (b) What is the height of the point *P* at the tip of a blade at 5s? 40s?
- (c) At what time is the point *P* exactly 7 m above the ground?

(C) Writing Sinusoidal Equations from Word Problems

- ex 6. In the Bay of Fundy, the depth of water around a dock changes from low tide around 03:00 to high tide at 09:00. The data shown below shows the water depth in a 24 hour period

Time (h)	0	3	6	9	12	15	18	21	24
Depth (m)	8.4	1.5	8.3	15.6	8.5	1.6	8.4	15.4	8.5

- (a) Prepare a scatter plot of the data and draw the curve of best fit
- (b) Determine an equation of the curve of best fit
- (c) You can enter the data into a GC and do a SinReg to determine the curve of best fit
- (d) Compare your equation to the calculator=s equation.
- (e) Will it be safe for a boat to enter the harbour between 15:00 and 16:00 if it requires at least 3.5 m of water? Explain and confirm will algebraic calculation.

(D) Homework

- Nelson text, page 464, Q8,9,10,12,13-19