Using Series to Analyze Financial Situations: Future Value

2.7

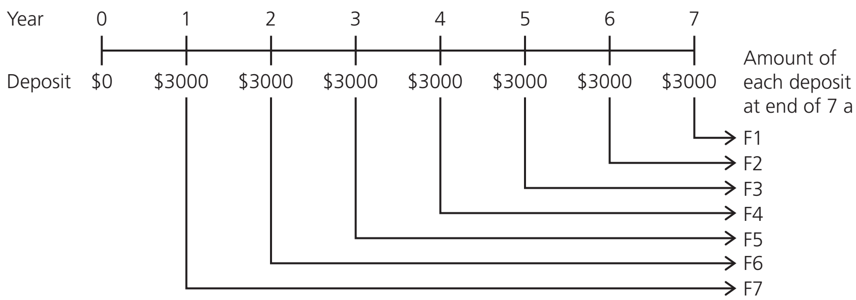
In section 2.5, you represented the future value of an ordinary simple annuity by finding the new balance after each payment and then adding the new balances. The amount of the annuity is the accumulated value of all the periodic payments, including interest.

In this section, you will investigate a method for finding the amount, or accumulated value, of an annuity by applying your knowledge of series.

Part 1: Representing an Ordinary Simple Annuity with a Series

In section 2.5, Mr. Watts wanted to help pay for his 12-year-old granddaughter's future university expenses. Mr. Watts decided to deposit $3000 at the end of each year, for seven years, in a savings account that pays 7.5%/a, compounded annually. Find the amount of the annuity at the end of seven years.

The amount of each deposit is shown in the **time line.** This time line shows the seven periods and all of the periodic payments. Each symbol, from F1 to F7, represents the amount, or accumulated value, of **each** deposit at the end of seven years.



Think, Do, Discuss

1. Each of the symbols from F1 to F7 represents the amount, or future value, of the payment at the end of seven years. What is the amount of the last deposit, F1?

2. At the end of the sixth year, Mr. Watts deposits $3000. For how many periods is this deposit in the account? Write an expression to represent F2, the future value of the deposit made at the end of the sixth year.

3. Repeat step 2 for each of the other deposits, F3 to F7.

4. Copy and complete the time line.

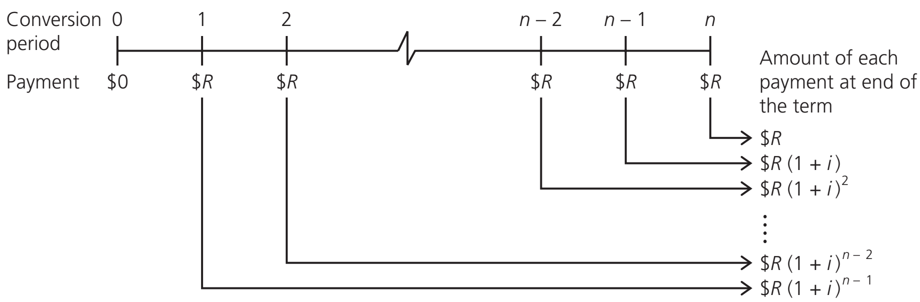
5. Write the series that represents the future values or amounts of all the deposits at the end of seven years. Use the expressions that you found in steps 1 to 3. Begin with F1.

6. Describe this type of series. What is the first term? Describe the first term in the context of the investment. What is the common ratio? Describe the common ratio in the context of the investment. How many terms are in the series?

7. Recall the formula for the sum of this type of series. Find the sum of the future values of the deposits. Compare your solution to the one you found in the Think, Do, Discuss of Part 1 in section 2.5.

Part 2: Developing a Formula for the Amount of an Ordinary Simple Annuity

Andrea wants to find an algebraic expression to represent the amount, or accumulated value, of an ordinary simple annuity. She lets R be the regular periodic payment of the annuity and lets n be the total number of interest conversion periods, or the total number of payments. She drew the following time line.



Think, Do, Discuss

1. What does i represent in the time line? Explain why the exponent in the future value of the first payment is $n - 1$.

2. Given Andrea's time line, write the series of terms, S_n, that represents the future value or amount of each payment of the annuity, beginning with the nth payment. What type of series is S_n?

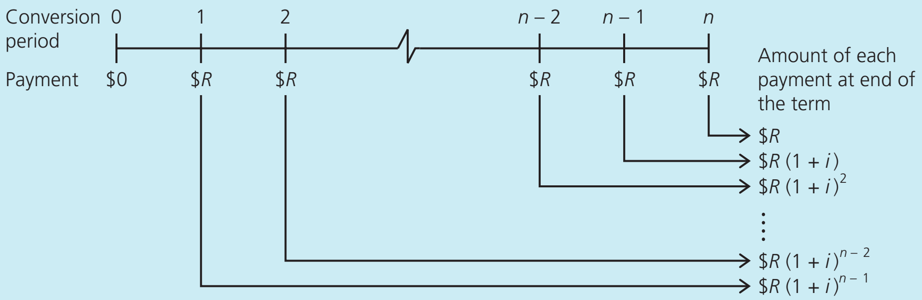
3. What is the first term of the series? What is the common ratio? Substitute these values in the formula you used in step 7 of Part 1. Simplify the denominator.

4. Use this formula again to find the amount of an annuity if the amount of each payment is $1000, payable at the end of every six months for three years. The interest rate is 10%/a, compounded semiannually.

Focus 2.7

Key Ideas

- The **amount**, or accumulated value, of an annuity is the sum of all the future values of all payments at the end of the term of the annuity.
- Show the amount of an ordinary simple annuity in a **time line**. A time line includes the payment intervals, the periodic payments or deposits, and the amount of each payment or deposit at the end of the term.



- The **amount** — the accumulated value or the future value — of an ordinary simple annuity can be written as the geometric series

$$R + R(1 + i) + R(1 + i)^2 + \ldots + R(1 + i)^{n-2} + R(1 + i)^{n-1}$$

where R represents the regular payment

n represents the number of interest conversion periods or the total number of payments

i represents the interest rate per conversion period

- An accumulated or future value of an ordinary simple annuity is the sum of a geometric series, $S_n = \frac{a(r^n - 1)}{r - 1}$, where $a = R$, $r = 1 + i$, and n is the number of payments.
- Another formula for calculating the accumulated or future value of an ordinary simple annuity, S_n or FV, is

$$\begin{aligned} FV &= S_n \\ &= R \times \frac{(1 + i)^n - 1}{i}. \end{aligned}$$

Example 1

For ten years, Shelagh deposits \$750 at the end of every three months in a savings account that pays 8%/a, compounded quarterly.

(a) Draw a time line to represent the annuity.

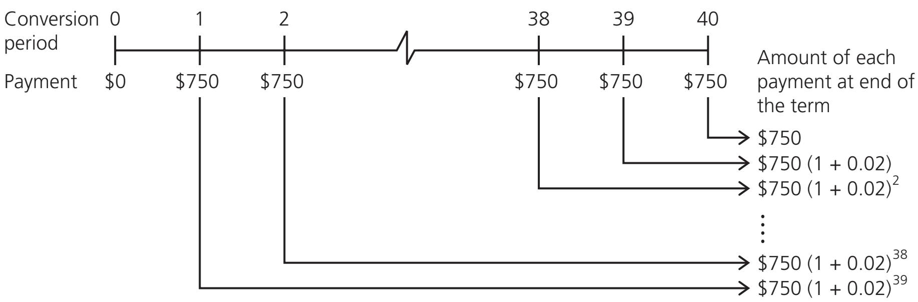
(b) Write the series that represents the annuity.

(c) Find the amount of the annuity and the total interest earned.

(d) Verify your results using **sequence** (from the List OPS menu) and **sum** (from the List MATH menu) on the TI-83 Plus calculator.

Solution

(a) The quarterly interest rate is 2% = 0.02 and the number of periods is $4 \times 10 = 40$.



(b) The series is $S_{40} = 750 + 750(1 + 0.02) + 750(1 + 0.02)^2 + \ldots + 750(1 + 0.02)^{38} + 750(1 + 0.02)^{39}$.

(c) **Method 1:** Using $S_n = \frac{a(r^n - 1)}{r - 1}$

The series is geometric, and $a = 750$, $r = 1.02$, and $n = 40$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Substitute the values for a, r, and n.

$$S_n = \frac{750(1.02^{40} - 1)}{1.02 - 1} \qquad \text{Simplify.}$$
$$= 45\ 301.49$$

The amount of the annuity is \$45 301.49.

Find the total interest earned on the annuity by subtracting the total payments from the accumulated value.

$$\text{total interest} = \$45\ 301.49 - (40 \times \$750)$$
$$= \$15\ 301.49$$

Method 2: Using $FV = R \times \frac{(1 + i)^n - 1}{i}$

The series represents the future value or amount of an ordinary simple annuity, where $R = 750$, $i = 0.02$, and $n = 40$.

$$FV = R \times \frac{(1 + i)^n - 1}{i} \quad \text{Substitute the values for } R, i, \text{ and } n.$$

$$= 750 \times \frac{(1.02)^{40} - 1}{0.02} \quad \text{Simplify.}$$

$$= 45\ 301.49$$

The total interest earned is \$45 301.49, as found in Method 1.

(d) The series is the first 40 terms of a geometric sequence with the general term $t_n = 750(1.02)^{n-1}$. The sum of the sequence is \$45 301.49.

Example 2

Roberta is an electrical engineer. She hopes to retire at 55, with savings. She plans to make equal monthly deposits, at the end of each month, for 20 years in a trust account that has a guaranteed interest rate of 9%/a, compounded monthly. She wants to have \$300 000 in the account at the end of 20 years.

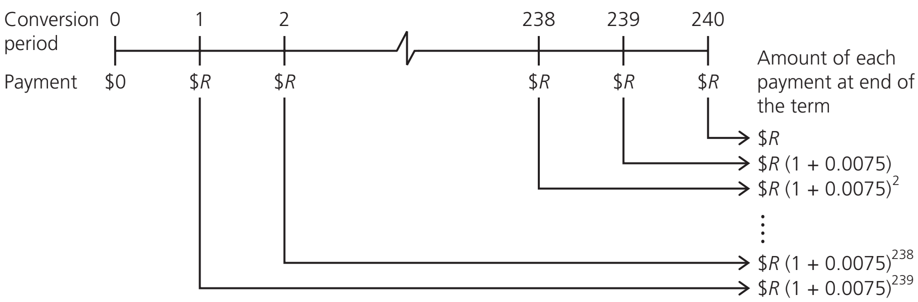
(a) Draw a time line to represent the annuity.

(b) Write the series that represents the annuity.

(c) What amount must be her monthly deposit?

Solution

(a) Find the monthly payment, R. The accumulated value is \$300 000, the interest rate per conversion period, i, is 0.0075, and the total number of periods, n, is 12×20 or 240.



(b) The series is $S_{240} = R + R(1 + 0.0075) + R(1 + 0.0075)^2 + \ldots + R(1 + 0.0075)^{238} + R(1 + 0.0075)^{239}$ and $S_{240} = 300\ 000$.

(c) Use $S_{240} = \frac{a(r^{240} - 1)}{r - 1}$, where $a = R$, $r = 1.0075$, and $S_{240} = 300\ 000$, or use $FV = R \times \frac{(1 + i)^n - 1}{i}$, where $n = 240$, $i = 0.0075$, and $FV = 300\ 000$.

Solve for R.

$$300\ 000 = R \times \left(\frac{(1.0075)^{240} - 1}{0.0075}\right)$$

Evaluate the expression in brackets.

$$300\ 000 \doteq R(667.886\ 869\ 9) \quad \text{Solve for } R.$$
$$449.18 \doteq R$$

Roberta must deposit \$449.18 each month.

Example 3

Rebecca has just celebrated her 12th birthday. She decides to save her \$100 monthly allowance and will deposit \$1200 at the end of each year in an account that pays 11%/a, compounded annually.

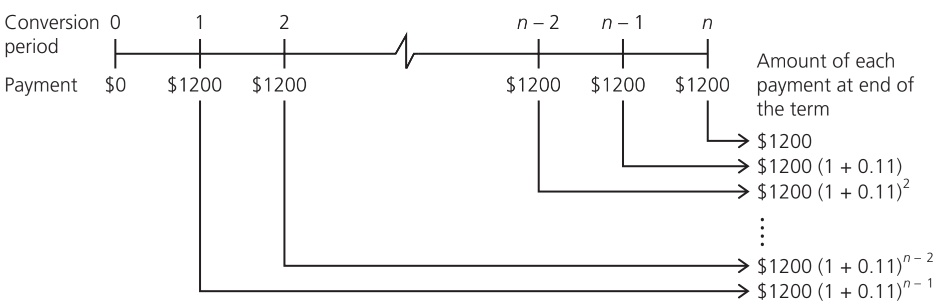
(a) Draw a time line to represent the annuity.

(b) Write the series that represents the annuity.

(c) Determine how old Rebecca will be when her annuity is worth \$20 000.

Solution

(a) The number of periods is unknown. The regular deposits are \$1200 each and the interest rate per conversion period is 11% = 0.11. The future value of the annuity must be \$20 000.



(b) The series is $S_n = 1200 + 1200(1.11) + 1200(1.11)^2 + \ldots + 1200(1.11)^{n-2} + 1200(1.11)^{n-1}$ and $S_n = 20\ 000$.

(c) Find n. $R = 1200$, $i = 0.11$, and $FV = 20\ 000$

$$FV = R \times \frac{(1+i)^n - 1}{i}$$ Substitute the values for R, i, and FV.

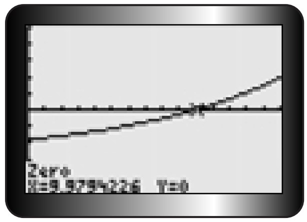
$$20\ 000 = 1200 \times \frac{(1.11)^n - 1}{0.11}$$ Simplify.

$$2200 = 1200\,(1.11^n - 1)$$ Divide by 1200.

$$1.833\ 33 \doteq 1.11^n - 1$$

$$2.833\ 33 \doteq 1.11^n$$

Solve this equation by graphing $y = 1.11^n - 2.83333$ and finding the zero.



Alternatively, you can determine that $1.11^{10} = 2.839\ 42$ and that $1.11^{9.97} = 2.830\ 54$, by using trial and error.

It will take just less than ten years for her deposits to be worth \$20 000. At that time, Rebecca will be about to celebrate her 22nd birthday.

Practise, Apply, Solve 2.7

A

1. Calculate each of the following. Express your answers to six decimal places.

(a) $(1.08)^{10}$ (b) $(1.12)^6$ (c) $(1.03)^{24}$

(d) $(1.005)^{60}$ (e) $(1.075)^{20}$ (f) $(1.05)^{48}$

2. Calculate each of the following. Express your answers to two decimal places.

(a) $3500 \times \frac{(1.09)^{10} - 1}{0.09}$ (b) $225 \times \frac{(1.06)^{24} - 1}{0.06}$

(c) $775 \times \frac{(1.001)^{60} - 1}{0.001}$ (d) $3000 \times \frac{(1.075)^{48} - 1}{0.075}$

3. Find, i, the interest rate per conversion period and, n, the total number of conversion periods for each of the following.

(a) The interest rate is 12%/a, compounded quarterly. The term is 3 years.

(b) The interest rate is 6.5%/a, compounded annually. The term is 10 years.

(c) The interest rate is 18%/a, compounded monthly. The term is 7 years.

(d) The interest rate is 13%/a, compounded weekly. The term is 5 years.

4. Evaluate.

(a) $100 + 100(1.07) + 100(1.07)^2 + \dots + 100(1.07)^{11} + 100(1.07)^{12}$

(b) $3200 + 3200(1.055) + 3200(1.055)^2 + \dots + 3200(1.055)^{23} + 3200(1.055)^{24}$

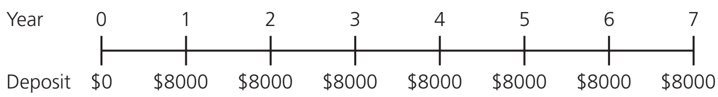
(c) $300 + 300(1.02) + 300(1.02)^2 + \dots + 300(1.02)^{39} + 300(1.02)^{40}$

5. Verify your solutions in question 4 using **sequence** (from the List OPS menu) and **sum** (from the List MATH menu) on the TI-83 Plus calculator.

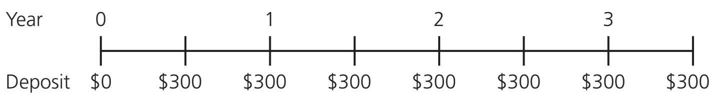
B

6. For each annuity
   i. find the amount of each payment at the end of the term
   ii. write the future or accumulated values of the payments as a series
   iii. find the accumulated value of the annuity

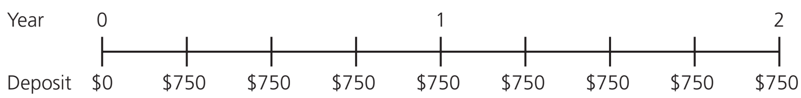
   (a) The rate of interest is 9%/a, compounded annually.



   (b) The rate of interest is 8%/a, compounded semiannually.



   (c) The rate of interest is 8%/a, compounded quarterly.



7. **Knowledge and Understanding:** Draw a time line to represent the amount of the annuity, where $1600 is deposited at the end of every three months for four years in an account that pays 10%/a, compounded quarterly.

8. For each annuity,
   i. draw a time line to represent the amount of the annuity
   ii. write the series that represents the amount of the annuity
   iii. find the amount of each annuity on the date of the last payment

   (a) $5000 is deposited at the end of every year for 10 years at 5%/a, compounded annually

   (b) $750 is deposited at the end of every 3 months for 5 years at 8%/a, compounded quarterly

   (c) $50 is deposited at the end of every week for 2 years at 13%/a, compounded weekly

   (d) $4300 is deposited at the end of every 6 months for 7 years at 9.5%/a, compounded semiannually

9. **(a)** Draw a time line and find the amount of each annuity in question 8 if the periodic payment is made at the beginning of each payment interval.

(b) Verify your solutions using technology.

10. Tara and Tobey are planning to buy a home in three years. They would like to accumulate enough money for a \$30 000 down payment. They will deposit the same sum of money at the end of each month in an account that pays 6%/a, compounded monthly.

(a) Draw a time line to represent the annuity.

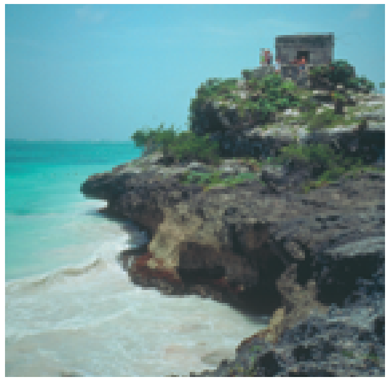
(b) Write the series that represents the annuity.

(c) Determine the monthly deposit required to meet their goal.

11. At the end of every six months, Marcia deposited \$100 in a savings account that paid 4%/a, compounded semiannually. She made the first deposit when her son was six months old and she made the last deposit on her son's 21st birthday. The money remained in the account until her son turned 25, when Marcia gave him the money. How much did he receive?

12. At the end of each month, \$50 is deposited in an account for 25 years. Then the accumulated money remains in the account for an additional 5 years. Find the amount in the account at the end of this time. The interest rate is 4.8%/a, compounded monthly.

13. Marcel would like to take a vacation to Mexico during March break, which is nine months from today. The trip will cost \$1600. Marcel saves and deposits \$195 at the end of each month for the next eight months at 9%/a, compounded monthly. Will he have enough money to pay for his trip?



14. **Communication:** Explain why the two formulas for finding the accumulated value of a simple annuity would not work if the interest conversion period did not coincide with the payment interval.

15. During the school year, Quentin has a part-time job working at the library. He deposits \$75 at the end of each month for three years in an education fund that pays 6%/a, compounded monthly.

(a) Draw a time line to represent the annuity.

(b) Write the series that represents the annuity.

(c) Determine how much is in the fund after three years.

(d) Verify your solution using technology. Graph the monthly balance over the three years.

(e) Determine how many payments are needed so that \$5200 will accumulate in the fund.

16. Rhys deposits \$2000 at the end of every three months in an account that pays 6%/a, compounded quarterly.

(**a**) Determine how many payments he will need to make so that the value of the account is worth at least \$45 000.

(**b**) Use technology to verify your solution and to graph the quarterly balances up to a value of \$45 000.

17. Samantha has just celebrated her seventh birthday. Her aunt Lise decides to start an education fund in Samantha's name and deposits \$450 at the end of every three months in a fund that pays 8.4%/a, compounded quarterly.

(**a**) Draw a time line to represent the annuity.

(**b**) Write the series that represents the annuity.

(**c**) Determine how old Samantha will be when the fund is worth \$24 000.

(**d**) How much less time would it take for the fund to accumulate to \$24 000 if Lise made deposits of \$530 each?

18. **Application:** Mario deposits \$25 at the end of each month for four years in an account that pays 9.6%/a, compounded monthly. He then makes no further deposits and no withdrawals. Find the balance after ten years.

19. Darcey is a pastry chef. He would like to accumulate \$80 000 in savings before he retires 20 years from now. He intends to make the same deposit at the end of each month in a registered retirement savings plan that pays 6.3%/a, compounded monthly.

(**a**) Draw a time line to represent the annuity.

(**b**) Write the series that represents the annuity.

(**c**) Determine the periodic payment that will enable him to reach his goal.

(**d**) He decides to wait five years before making the monthly deposits. What periodic payment would he have to make to reach the same goal?

(**e**) About how much more money did he have to invest by waiting five years?

20. Gwyneth has \$4320 in her savings account and she deposits \$800 at the end of every three months. The account earns 6%/a, compounded quarterly. What will be the balance in the account after five years?

21. **Thinking, Inquiry, Problem Solving:** Eire was wondering which of these options is better — to double the interest of an annuity, or to double the term of an annuity. Work with a partner to determine if one option is better than the other and explain your findings.

22. **Check Your Understanding**

(**a**) In Example 2, Roberta made monthly deposits for 20 years to attain her financial goal. How would her monthly deposit be affected if she had started making deposits five years earlier, that is, if she made deposits for 25 years? Explain.

(**b**) Find the periodic payment, if Roberta had started making deposits five years earlier. How much less would she have had to pay?

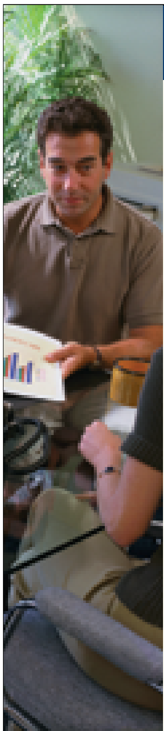
C

23. Fatuma has decided to invest $3000 at the end of each year for the next five years in a savings account that pays 8%/a, compounded semiannually.

(a) Draw a time line to represent the accumulated value of each payment.

(b) Determine the amount of the annuity.

24. Mr. Friday is preparing his will. He wants to leave the same amount of money to his son and to his daughter. His daughter is careful with money, but his son spends it carelessly, so he decides to give them the money in different ways. The interest rate is 6%/a, compounded monthly. How much must his estate pay his son each month over 20 years so that the accumulated value will be equal to the $50 000 cash his daughter will receive upon his death? Assume that the daughter's inheritance earns the same interest rate over the 20 years.



The Chapter Problem—Financial Planning

In this section, you studied future value. Apply what you learned to answer these questions about the Chapter Problem on page 106.

CP10. Confirm the accumulated value of Bart's education fund on his 18th birthday, including the payment on his 18th birthday. Use the formula for the accumulated value of an ordinary simple annuity.

CP11. Can you determine the periodic payment for the registered income fund (RIF) given what you know about the accumulated value of an annuity? Explain.

CP12. Mr. Sacchetto intends to make a deposit at the end of every three months in a savings account that earns 8%/a, compounded quarterly, to meet his retirement goal of having $120 000 in savings at age 55.

(a) Draw a time line to represent the accumulated values of Mr. Sacchetto's deposits in the savings account.

(b) List the series that represents the accumulated values of the RIF.

(c) For his savings account, determine the quarterly deposit that Mr. Sacchetto must make to meet his retirement goal.

(d) Determine how much more money in total he would have to deposit if the account pays only 6%/a, compounded quarterly.