Lesson 25 - Applications of Logarithmic Functions

Work through the following word problems, presenting complete algebraic & graphic solutions to the questions.

1. Mr. S. drinks a cup of coffee at 9:45 am and his coffee contains 150 mg of caffeine. Since the half-life of caffeine for an average adult is 5.5 hours,
   a. determine how much caffeine is in Mr. S.'s body at class-time (1:10pm).
      $A(3.41\dot{6}) = 150(2)^{-\left(\frac{3.41\dot{6}}{5.5}\right)} = 97.5mg$
   b. Then determine how much time passes before I have 30 mg of caffeine in my body
      $A(t) = 30 = 150(2)^{-\left(\frac{t}{5.5}\right)} \Rightarrow t = 12.8hrs$

2. Two populations of bacteria are growing at different rates. Their populations at time *t* are given by $P_1(t) = P_o5^{t+2}$ and $P_2(t) = P_oe^{rt}$ respectively.
   a. At what time are the populations the same? $P_1(t) = P_2(t) \Rightarrow P_05^{t+2} = P_0e^{2t} \Rightarrow t = 8.24$
   b. When is the population of P_2 twice that of P_1? $2P_1(t) = P_2(t) \Rightarrow 2P_05^{t+2} = P_0e^{2t} \Rightarrow t = 10.02$

3. Dry cleaners use a cleaning fluid that is purified by evaporation and condensation after each cleaning cycle. Every time it is purified, 2% of the fluid is lost . ➔ HINT: Eqn to use ➔ $A(n) = A_0(1-0.02)^n$
   a. An equipment manufacturer claims that after 20 cycles, about two-thirds of the fluid remains. Verify or reject this claim. $A(20) = A_0(1-0.02)^{20} = A_0(0.66761...) \Rightarrow \therefore true$
   b. If the fluid has to be "topped up" when half the original amount remains, after how many cycles should the fluid be topped up? $A(n) = 0.50A_0 = A_0(1-0.02)^n \Rightarrow t = 34 \text{ or } 35 \text{ cycles}$
   c. A manufacturer has developed a new process such that two-thirds of the cleaning fluid remains after 40 cycles. What percentage of fluid is lost after each cycle? $A(40) = \frac{2}{3}A_0 = A_0(1-r)^{40} \Rightarrow r = 0.01$

4. You invest \$5000 in a stock that grows at a rate of 12% per annum compounded quarterly. The value of the stock is given by the equation $V = 5000(1 + 0.12/4)^{4x}$ or $V = 5000(1.03)^{4x}$ where x is measured in years.
   a. Find the value of the stock in 6 years. $V(6) = 5000(1.03)^{(4\times6)} = 10163.97$
   b. Find when the stock value is \$14,000 $V(x) = 14000 = 5000(1.03)^{(4x)} \Rightarrow x = 8.71$

5. The population of a small town was 35,000 in 1980 and in 1990, it was 57,010. ➔ HINT: Eqn to use ➔ $P(t) = 35{,}000(1+r)^t$
   a. Create an algebraic model for the town's population growth.
      $P(10) = 57{,}010 = 35{,}000(1+r)^{10} \Rightarrow r = 0.05$
   b. Check your model using the fact that the population was 72800 in 1995.
      $P(15) = 35{,}000(1+0.05)^{15} = 72762 \Rightarrow$ close to 72800
   c. What will the population be in 2010? $P(30) = 35{,}000(1+0.05)^{30} = 151267$
   d. When will the population be 100,000? $P(t) = 100{,}000 = 35{,}000(1+0.05)^t \Rightarrow t = 21.5$

6. The sales S (in thousands of units) of a new CD burner after it has been on the market for *t* years are given by S(t) = 100(1 – e^{kt}). Fifteen thousand units of the new product were sold the first year.

   a. Complete the model by solving for *k*. $S(1)=15=100\left(1-e^{k}\right)\Rightarrow k=-0.1625$

   b. Use your calculator to graph the model.

   c. Determine S(3) and interpret. $S(3)=100\left(1-e^{-0.1625}\right)=38.6$

   d. When were 8,000 units sold? $S(t)8=100\left(1-e^{-0.1625t}\right)\Rightarrow t=0.5y$

7. A conservation authority releases 100 animals of an endangered species into a game preserve. The organization believes that the preserve has a carrying capacity of 1,000 animals and that the growth of the herd will follow the logistic curve $p(t)=\dfrac{1000}{1+9e^{-0.1656t}}$, where t is time measured in months.

   a. Graph the function and determine the values of p at which the horizontal asymptotes occur. Interpret the meaning of the asymptotes in the context of the problem. P = 1000 (Carrying capacity)

   b. Estimate the population after 5 years. $p(5)=\dfrac{1000}{1+9e^{-0.1656\times5}}\approx 203$

   c. When will the population reach 500? $p(t)=500=\dfrac{1000}{1+9e^{-0.1656t}}\Rightarrow t=13.3$

8. A coroner is called in to investigate a death. The coroner determines the body's temperature at 9:00 to be 85.7°F and at 11:00 the temperature was 82.8°F. The relationship between time elapsed since death (*t*, in hours) and the body temperature (T in degrees Fahrenheit) is $t(T)=-10\ln\left(\dfrac{T-70}{98.6-70}\right)$. (assume that the body had a normal body temperature of 98.6°F and the room temperature was 70°F). Estimate the time of death of the body. $t(85.7)=-10\ln\left(\dfrac{85.7-70}{98.6-70}\right)\approx 6$ and $t(82.8)=-10\ln\left(\dfrac{82.8-70}{98.6-70}\right)\approx 8$ so 6 hours BEFORE 9:00 or 8 hours before 11:00, so at 03:00 am

9. The model $L(p)=12.542\ln\left(\dfrac{p}{p-1000}\right), p>1000$ approximates the length of a home mortgage of $150,000 at 8% in terms of the monthly payment. In the model, *L* is the length of the mortgage in years and *p* is the monthly payments in dollars.

   a. Use the model to approximate the length of the mortgage when the monthly payments are $1254.68

   $$L(1254.68)=12.542\ln\left(\frac{1254.68}{1254.68-1000}\right)\approx 20$$

   b. Approximate the total amount paid over the term of the mortgage from (a). What amount of the total is interest costs? $20\times12\times1254.68=301123.2\Rightarrow\therefore 301123.2-150{,}000=151{,}123.2$

   c. If I want to pay off the mortgage in 17 years, how much should my monthly payments be?

   $$L(p)=17=12.542\ln\left(\frac{p}{p-1000}\right)\Rightarrow p=1347.40$$

10. In a psychology project on learning theory, a mathematical model for the proportion of correct responses, *P*, after *n* trials was found to be $P(n)=\frac{0.83}{1+e^{-0.2n}}$.

   a. Graph the function
   b. Determine the equation(s) of any horizontal asymptotes and interpret in the context of the problem.
   $P=0.83 \Rightarrow \text{max \% correct will be 83\% regardless of \# of trials}$

   c. Determine *P*(0) and *P*(12) and interpret.
   $P(0)=\frac{0.83}{1+e^{0}}=41.5\% \quad \text{you will initially get 41\% of the responses correct}$
   $P(12)=\frac{0.83}{1+e^{-0.2\times 12}}=76.1\% \quad \text{you will get 76.1\% of the responses correct after you have had 12 trials}$

   d. After how many trials will 60% of the responses be correct? $P(n)=0.60=\frac{0.83}{1+e^{-0.2n}} \Rightarrow n\approx 4.8$

11. The demand function for a camera is given by $p(x)=500-0.5e^{0.004x}$, where *p* is the price of the camera in dollars and *x* is the demand (how many units can be sold at that price).

   a. Determine *p*(1600) and interpret in the context of the problem.
   $p(1600)=500-0.5e^{0.004\times 1600}=199 \Rightarrow \text{ if the price is \$199, then 1600 cameras will be sold}$

   b. Determine the demand, x , for a price of
      i. \$600 $p(x)=600=500-0.5e^{0.004x}$
      $\Rightarrow \text{x has NO solution as the price at \$600 is TOO high for a \$500, so NO units sold}$
      ii. \$400 $p(x)=400=500-0.5e^{0.004x} \Rightarrow x=1324$