Lesson 20 – Introducing and Applying Base *e*.

IB Math SL1 - Santowski

Lesson Objectives

- (1) Investigate a new base to use in exponential applications
- (2) Understand WHEN the use of base e is appropriate
- (3) Apply base e in word problems and equations

Lesson Objective #1

- (1) Investigate a new base to use in exponential applications

- GIVEN: the formula for working with compound interest ➔

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

- Determine the value after 2 years of a $1000 investment under the following compounding conditions:

(A) Working with Compounding Interest

- Determine the value after 2 years of a $1000 investment under the following investing conditions:

- (a) Simple interest of 10% p.a
- (b) Compound interest of 10% pa compounded annually
- (c) Compound interest of 10% pa compounded semi-annually
- (d) Compound interest of 10% pa compounded quarterly
- (e) Compound interest of 10% pa compounded daily
- (f) Compound interest of 10% pa compounded n times per year

(B) Introducing Base *e*

- Take $1000 and let it grow at a rate of 10% p.a. Then determine value of the $1000 after 2 years under the following compounding conditions:
- (i) compounded annually $=1000(1 + .1/1)^{(2x1)} = 1210$
- (ii) compounded quarterly $=1000(1 + 0.1/4)^{(2x4)} = 1218.40$
- (iii) compounded daily $=1000(1 + 0.1/365)^{(2x365)} = 1221.37$
- (iv) compounded *n* times per year $=1000(1+0.1/n)^{(2xn)} = 1210$

(B) Introducing Base e

- So we have the expression $1000(1 + 0.1/n)^{(2xn)}$

- Now what happens as we increase the number of times we compound per annum $\Rightarrow$ i.e. $n \rightarrow \infty$?? (that is … come to the point of compounding continuously)

- So we get a limit: $\lim_{n \to \infty} \left(1000 \times \left(1 + \frac{0.1}{n} \right)^{(2 \times n)} \right)$

(B) Introducing Base e

- Now let's rearrange our limit
- use a simple substitution ➔ let 0.1/n = 1/x
- Therefore, 0.1x = n ➔ so then $\lim_{n\to\infty}\left(1000\times\left(1+\frac{0.1}{n}\right)^{(2\times n)}\right)$
- becomes $\lim_{x\to\infty}\left(1000\times\left(1+\frac{1}{x}\right)^{(x\times 0.1\times 2)}\right)$
- Which simplifies to $1000\times\lim_{x\to\infty}\left(\left(1+\frac{1}{x}\right)^{x}\right)^{0.1\times 2}$

(B) Introducing Base *e*

- So we see a special limit occurring: $\lim_{x\to\infty}\left(\left(1+\frac{1}{x}\right)^{x}\right)$

- We can evaluate the limit a number of ways ➔ graphing or a table of values.

(B) Introducing Base *e*

- So we see a special limit occurring: $\lim_{x\to\infty}\left(\left(1+\frac{1}{x}\right)^{x}\right)$

- We can evaluate the limit a number of ways ➔ graphing or a table of values.

- In either case, $\lim_{x\to\infty}\left(\left(1+\frac{1}{x}\right)^{x}\right)=e$ where *e* is the natural base of the exponential function

(B) Introducing Base *e*

- So our original formula $\lim_{n\to\infty}\left(1000\times\left(1+\frac{0.1}{n}\right)^{(2\times n)}\right)$ now becomes A = $1000e^{0.1x2}$ where the 0.1 was the interest rate, 2 was the length of the investment (2 years) and $1000 was the original investment (so A = Pe^{rt}) ➔ so our value becomes $1221.40

- And our general equation can be written as A = Pe^{rt} where P is the original amount, r is the annual growth rate and t is the length of time in years

Lesson Objective #2

- (2) Understand WHEN the use of base e is appropriate

- Recall our original question ➔ Determine the value after 2 years of a $1000 investment under the following investing conditions:

- (a) Compound interest of 10% pa compounded annually
- (b) Compound interest of 10% pa compounded semi-annually
- (c) Compound interest of 10% pa compounded quarterly
- (d) Compound interest of 10% pa compounded daily

- All these examples illustrate **DISCRETE** changes rather than **CONTINUOUS** changes.

Lesson Objective #2

- (2) Understand WHEN the use of base e is appropriate

- So our original formula $\lim_{n\to\infty}\left(1000\times\left(1+\frac{0.1}{n}\right)^{(2\times n)}\right)$ now becomes A = $1000e^{0.1x2}$ where the 0.1 was the annual interest rate, 2 was the length of the investment (2 years) and $1000 was the original investment ➔ BUT RECALL WHY we use "n" and what it represents …….

- Note that in this example, the growth happens ***<u>continuously</u>*** (i.e the idea that n→∞)

Lesson Objective #3

- (3) Apply base ***e*** in word problems and equations

(C) Working With Exponential Equations in Base **e**

- (i) Solve the following equations:

$$(i)\ \ e^{x^2-x}=e^2 \qquad (ii)\ \ \left(e^x\right)^2=\sqrt{e^{x+2}}$$

$$(ii)\ \ e^{-x^2}=\left(\frac{1}{e}\right)^x \qquad (iv)\ e^{2x-1}=\frac{1}{e^{3x+1}}$$

$$(v)\ \ e^x=4 \qquad (vi)\ e^x=-5$$

$$(vii)\ \ e^x=1-x \qquad (viii)\ e^{2x}+6=5e^x$$

(C) Working with $A = Pe^{rt}$

- So our formula for situations featuring continuous change becomes $A = Pe^{rt}$ ➔ P represents an initial amount, r the annual growth/decay rate and t the number of years

- In the formula, if $r > 0$, we have exponential growth and if $r < 0$, we have exponential decay

(C) Examples

- (i) I invest $10,000 in a funding yielding 12% p.a. compounded continuously.
  - (a) Find the value of the investment after 5 years.
  - (b) How long does it take for the investment to triple in value?
- (ii) The population of the USA can be modeled by the eqn $P(t) = 227e^{0.0093t}$, where P is population in millions and t is time in years since 1980
  - (a) What is the annual growth rate?
  - (b) What is the predicted population in 2015?
  - (c) What assumptions are being made in question (b)?
  - (d) When will the population reach 500 million?

(C) Examples

- (iii) A certain bacteria grows according to the formula $A(t) = 5000e^{0.4055t}$, where t is time in hours.
  - (a) What will the population be in 8 hours
  - (b) When will the population reach 1,000,000

- (iv) The function $P(t) = 1 - e^{-0.0479t}$ gives the percentage of the population that has seen a new TV show t weeks after it goes on the air.
  - (a) What percentage of people have seen the show after 24 weeks?
  - (b) Approximately, when will 90% of the people have seen the show?
  - (c) What happens to P(t) as t gets infinitely large? Why? Is this reasonable?

Homework

- IB Textbook:

- Ex5A, p106-7, Q5-7,8bcd,9egh,10-13,
- Ex5E, p113-4, Q1-5 ➔ Solve graphically