Lesson 16 – Modeling with Exponential Functions

Math SL - Santowski

Comparing Linear, Quadratic & Exponential Models

- Data set #1

X	0	1	2	3	4	5	6
y	50	75	112.5	168.75	253.13	379.69	569.53

- Data set #2

X	0	1	2	3	4	5	6
y	5	7.75	10.5	13.25	16	18.75	21.5

- Data set #3

X	0	1	2	3	4	5	6
y	3	6	11	18	27	38	51

Exponential Models

- Key Data feature ➔ common RATIO between consecutive terms
- Consider $f(x) = 2^x$ ➔
- 1. create a table of values ➔ pattern to observe …
- 2. Create a graph
- 3. function value at x = 5 is …..
- 4. Function value at x = 3.25 is ….
- 5. The rate of increase is …..

Working with Exponential Models

- Populations can also grow exponentially according to the formula $P = P_o(1.0125)^n$. If a population of 4,000,000 people grows according to this formula, determine:
- 1. the population after 5 years
- 2. the population after 12.25 years
- 3. when will the population be 6,500,000
- 4. what is the average annual rate of increase of the population

Working with Exponential Models

- The value of a car depreciates according to the exponential equation $V(t) = 25{,}000(0.8)^t$, where t is time measured in years since the car's purchase. Determine:
- 1. the car's value after 5 years
- 2. the car's value after 7.5 years
- 3. when will the car's value be $8,000
- 4. what is the average annual rate of decrease of the car's value?

Working with Exponential Models

- Bacteria grow exponentially according to the equation $N(t) = N_o 2^{(t/d)}$ where N(t) is the amount after a certain time period, N_o is the initial amount, t is the time and d is the doubling period
- A bacterial strain doubles every 30 minutes. If the starting population was 4,000, determine
- 1. the number of bacteria after 85 minutes
- 2. the number of bacteria after 5 hours
- 3. when will the number of bacteria be 30,000
- 4. what is the average hourly rate of increase of the # of bacteria?
- 4. what is the average daily rate of increase of the # of bacteria?

Working with Exponential Models

- Radioactive chemicals decay over time according to the formula $N(t) = N_o 2^{(-t/h)}$ where N(t) is the amount after a certain time period, N_o is the initial amount, t is the time and h is the halving time → which we can rewrite as $N(t) = N_o(1/2)^{(t/h)}$ and the ratio of t/h is the number of halving periods.
- Ex 1. 320 mg of iodine-131 is stored in a lab but it half life is 60 days. Determine
- 1. the amount of I-131 left after 10 days
- 2. the amount of I-131 left after 180 days
- 3. when will the amount of I-131 left be 100 mg?
- 4. what is the average daily rate of decrease of I-131?
- 4. what is the average monthly rate of decrease of I-131?

(A) Modeling Example #1

- The following data table shows the relationship between the time (in hours after a rain storm in Manila) and the number of bacteria (#/mL of water) in water samples from the Pasig River:

Time (hours)	# of Bacteria
0	100
1	196
2	395
3	806
4	1570
5	3154
6	6215
7	12600
8	25300

(A) Modeling Example #1

- (a) Graph the data on a scatter plot
- (b) How do you know the data is exponential rather than quadratic?
- (c) How can you analyze the numeric data (no graphs) to conclude that the data is exponential?
- (d) Write an equation to model the data. Define your variables carefully.

(A) Modeling Example #2

- The value of Mr S car is depreciating over time. I bought the car new in 2002 and the value of my car (in thousands) over the years has been tabulated below:

Year	2002	2003	2004	2005	2006	2007	2008	2009	2010
Value	40	36	32.4	29.2	26.2	23.6	21.3	19.1	17.2

- (a) Graph the data on a scatter plot
- (b) How do you know the data is exponential rather than quadratic?
- (c) How can you analyze the numeric data (no graphs) to conclude that the data is exponential?
- (d) Write an equation to model the data. Define your variables carefully.

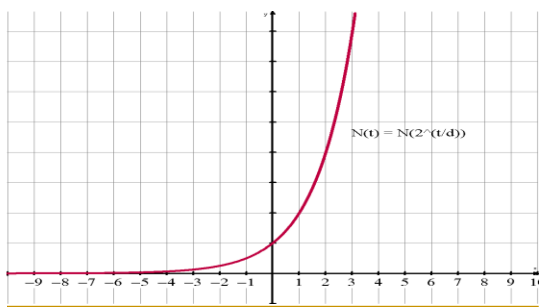
(A) Modeling Example #3

- The following data table shows the historic world population since 1950:

Year	1950	1960	1970	1980	1990	1995	2000	2005	2010
Pop (in millions)	2.56	3.04	3.71	4.45	5.29	5.780	6.09	6.47	6.85

- (a) Graph the data on a scatter plot
- (b) How do you know the data is exponential rather than quadratic?
- (c) How can you analyze the numeric data (no graphs) to conclude that the data is exponential?
- (d) Write an equation to model the data. Define your variables carefully.

(B) Exponential Growth Curve



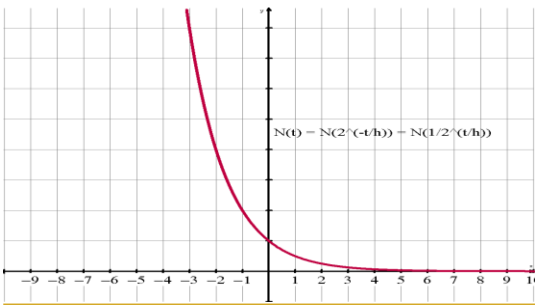
(C) Exponential Modeling

- In general, however, the algebraic model for exponential growth is $y = c(a)^x$ where a is referred to as the growth rate (provided that $a > 1$) and c is the initial amount present and x is the number of increases given the growth rate conditions.
- All equations in this section are also written in the form $y = c(1 + r)^x$ where c is a constant, r is a positive rate of change and $1 + r > 1$, and x is the number of increases given the growth rate conditions.

(D) Decay Curves

- certain radioactive chemicals like uranium have decay curves that can be characterized by exponential functions
- their decay is said to be exponential since they reduce by a ratio of two at regular intervals (their amount is half of what it was previously)
- to derive a formula for exponential decay, consider the following for a two hour halving period

(D) Exponential Decay Curve



(D) Exponential Decay Formula

- therefore, we come up with the formula $N(t) = N_o 2^{(-t/h)}$ where N(t) is the amount after a certain time period, N_o is the initial amount, t is the time and h is the halving time ➔ which we can rewrite as $N(t) = N_o(1/2)^{(t/h)}$ and the ratio of t/h is the number of halving periods.
- In general, however, the algebraic model for exponential decay is $y = c(a)^x$ where a is referred to as the decay rate (and a is < 1) and c is the initial amount present.
- All equations in this section are in the form $y = c(1 + r)^x$ or $y = ca^x$, where c is a constant, r is a rate of change (this time negative as we have a decrease so $1 + r < 1$), and x is the number of increases given the rate conditions

(F) Exponential Functions

- The features of the parent exponential function $y = a^x$ (where $a > 1$) are as follows:
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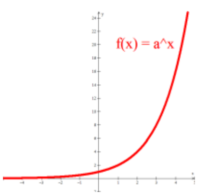

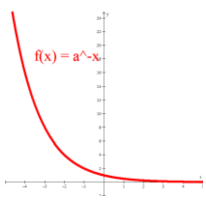

(F) Exponential Functions

- The features of the parent exponential function $y = a^x$ (where $a > 1$) are as follows:

- Domain ➔
- Range ➔
- Intercept ➔
- Increase on ➔ Asymptote ➔
- As $x \to -\infty$, $y \to$
- As $x \to \infty$, $y \to$

- The features of the parent exponential function $y = a^x$ (where $0 < a < 1$) are as follows:

- Domain ➔
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(G) Homework

- HW
- Ex 3G, Q2,3,4;
- Ex 3H #1,3,4;
- Ex 3I #1,3,4;

(B) Growth Curves

- certain organisms like bacteria and other unicellular organisms have growth curves that can be characterized by exponential functions
- their growth is said to be exponential since they duplicate at regular intervals
- we come up with the doubling formula $N(t) = N_o2^{(t/d)}$ where N(t) is the amount after a certain time period, N_o is the initial amount, t is the time and d is the doubling period

(C) Examples

- ex. 1 A bacterial strain doubles every 30 minutes. If there are 1,000 bacteria initially, how many are present after 6 hours?
- ex 2. The number of bacteria in a culture doubles every 2 hours. The population after 5 hours is 32,000. How many bacteria were there initially?

(C) Exponential Modeling

- Investments grow exponentially as well according to the formula $A = P_o(1 + i)^n$. If you invest $500 into an investment paying 7% interest compounded annually, what would be the total value of the investment after 5 years?
- (i) You invest $5000 in a stock that grows at a rate of 12% per annum compounded quarterly. The value of the stock is given by the equation $V = 5000(1 + 0.12/4)^{4x}$, or $V = 5000(1.03)^{4x}$ where x is measured in years.
  - (a) Find the value of the stock in 6 years.
  - (b) Find when the stock value is $14,000

(C) Examples

- ex. 4 Populations can also grow exponentially according to the formula $P = P_o(1 + r)^n$. If a population of 4,000,000 people grows at an average annual rate of increase of 1.25 %, find population increase after 25 years.
- ex 5. The population of a small town was 35,000 in 1980 and in 1990, it was 57,010. Create an algebraic model for the towns population growth. Check your model using the fact that the population was 72800 in 1995. What will the population be in 2010?

(E) Half Life - Examples

- Ex 1. 320 mg of iodine-131 is stored in a lab for 40d. At the end of this period, only 10 mg remains.
- (a) What is the half-life of I-131?
- (b) How much I-131 remains after 145 d?
- (c) When will the I-131 remaining be 0.125 mg?
- Ex 2. Health officials found traces of Radium F beneath P044. After 69 d, they noticed that a certain amount of the substance had decayed to $1/\sqrt{2}$ of its original mass. Determine the half-life of Radium F

(E) Examples

- ex 3. Three years ago there were 2500 fish in Loon Lake. Due to acid rain, there are now 1945 fish in the lake. Find the population 5 years from now, assuming exponential decay.
- ex 4. The value of a car depreciates by about 20% per year. Find the relative value of the car 6 years after it was purchased.
- Ex 5. When tap water is filtered through a layer of charcoal and other purifying agents, 30% of the impurities are removed. When the water is filtered through a second layer, again 30% of the remaining impurities are removed. How many layers are required to ensure that 97.5% of the impurities are removed from the tap water?