Example 2 Find $\frac{dy}{dx}$ if $y = \frac{\sqrt{x}}{1 + 2x}$.

Solution

$$\frac{dy}{dx} = \frac{(1 + 2x)\frac{d}{dx}\sqrt{x} - \sqrt{x}\frac{d}{dx}(1 + 2x)}{(1 + 2x)^2}$$

$$= \frac{(1 + 2x)\frac{1}{2\sqrt{x}} - \sqrt{x}(2)}{(1 + 2x)^2}$$

Now we multiply the numerator and denominator by $2\sqrt{x}$:

$$\frac{dy}{dx} = \frac{1 + 2x - (2\sqrt{x})(2\sqrt{x})}{2\sqrt{x}(1 + 2x)^2} = \frac{1 - 2x}{2\sqrt{x}(1 + 2x)^2}$$

EXERCISE 2.5

B **1.** Differentiate.

(a) $f(x) = \frac{x - 1}{x + 1}$ (b) $f(x) = \frac{2x - 1}{x^2 + 1}$

(c) $g(x) = \frac{x}{x^2 + 2x - 1}$ (d) $g(x) = \frac{x^3 - 1}{x^2 + x + 1}$

(e) $y = \frac{\sqrt{x}}{x^2 + 1}$ (f) $y = \frac{\sqrt{x} + 2}{\sqrt{x} - 2}$

(g) $f(t) = \frac{2t + 1}{t^2 - 3t + 4}$ (h) $g(t) = \frac{2t^2 + 3t + 1}{t - 1}$

(i) $f(x) = \frac{1}{x^4 - x^2 + 1}$ (j) $f(x) = \frac{ax + b}{cx + d}$

(k) $f(x) = \frac{x^6}{x^5 - 10}$ (l) $f(x) = \frac{1 - \frac{1}{x}}{x + 1}$

2. Find the domain of f and compute its derivative.

(a) $f(x) = \frac{2 + x}{1 - 2x}$ (b) $f(x) = \frac{x}{x^2 - 1}$

(c) $f(x) = \frac{1}{(x + 1)(2x - 3)}$ (d) $f(x) = \frac{2x + 1}{x^2 + 2x - 3}$

(e) $f(x) = \frac{x^2 + 2x}{x^4 - 1}$ (f) $f(x) = \frac{x^2}{\sqrt{x} - 3}$

3. Find an equation of the tangent line to the curve at the given point.

(a) $y = \dfrac{x}{x-2}$, $(4, 2)$ (b) $y = \dfrac{1+3x}{2-3x}$, $(1, -4)$

(c) $y = \dfrac{1}{x^2+1}$, $\left(-2, \frac{1}{5}\right)$ (d) $y = \dfrac{x^3-1}{1+2x^2}$, $(1, 0)$

4. If $f(2) = 3, f'(2) = 5$, $g(2) = -1$, and $g'(2) = -4$, find $\left(\dfrac{f}{g}\right)'(2)$.

5. Show that there are no tangents to the curve $y = \dfrac{x+2}{3x+4}$ with positive slope.

6. At what points on the curve $y = \dfrac{x^2}{2x+5}$ is the tangent line horizontal?

7. Find the points on the curve $y = \dfrac{x}{x-1}$ where the tangent line is parallel to the line $x + 4y = 1$.

8. If f is a differentiable function, find expressions for the derivatives of the following functions.

(a) $y = \dfrac{1}{f(x)}$ (b) $y = \dfrac{f(x)}{x}$ (c) $y = \dfrac{x}{f(x)}$

C **9.** In Section 2.2 we proved the Power Rule for positive integer exponents. Use the Quotient Rule to deduce the Power Rule for the case of negative integer exponents; that is, prove that

$$\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$$

when n is a positive integer.

2.6 THE CHAIN RULE

Although we have learned to differentiate a variety of functions, our differentiation rules still do not enable us to find the derivative of the function

$$F(x) = \sqrt{2x^2+3}$$

Notice that F is a composite function; it can be built up from simpler functions. If we let

$$y = f(u) = \sqrt{u} \quad \text{and} \quad u = g(x) = 2x^2 + 3$$

then $\quad f(g(x)) = f(2x^2+3) = \sqrt{2x^2+3} = F(x)$

(e) $f'(t) = (t^4 + t^2 - 1)(2t) + (t^2 - 2)(4t^3 + 2t)$
(f) $f'(t) = -\sqrt[3]{t} + \frac{1}{3}t^{-\frac{2}{3}}(1 - t)$
(g) $F'(y) = \sqrt{y}\left[1 - \frac{1}{\sqrt{y}}\right] + (y - 2\sqrt{y} + 2)\frac{1}{2\sqrt{y}}$
(h) $G'(y) = (y - y^2)(2 - \frac{4}{3}y^{\frac{1}{3}}) + (2y - y^{\frac{4}{3}})(1 - 2y)$

2. (a) $y' = 5x^4 + 8x^3 + 9x^2$
(b) $y' = 1 - 12x^{-3}$
(c) $f'(x) = 5x^4 - 3x^2 - 4x$
(d) $f'(x) = -36x^5 - 15x^2$
(e) $f'(t) = 480t^9 + 64t^7 - 90t^2 - 5$
(f) $f'(t) = 3act^2 + 2bct - ad$
(g) $g'(u) = \frac{45}{2}u^{\frac{7}{2}} - \frac{5}{2}u^{\frac{3}{2}} + u^{-\frac{1}{2}}$
(h) $g'(v) = 3v^2 - \frac{5}{2}v^{\frac{3}{2}} + \frac{3}{2}\sqrt{v} - 1$

3. (a) -13 (b) 0 (c) 20 (d) -9 (e) -11 (f) $-\frac{1}{2}$ **4.** 6

5. $x - y + 4 = 0$ **6.** -17

7. (a) $g'(x) = xf'(x) + f(x)$
(b) $h'(x) = \sqrt{x}f'(x) + \frac{1}{2\sqrt{x}}f(x)$
(c) $F'(x) = x^c f'(x) + cx^{c-1}f(x)$

8. (b) $y' = 2(2 + 5x - x^3)(5 - 3x^2)$

9. (b) $y' = \sqrt{x}\,(3x + 5)(12x - 5) + 3\sqrt{x}(6x^2 - 5x + 1) + \frac{1}{2\sqrt{x}}(3x + 5)(6x^2 - 5x + 1)$

10. (b) $y' = 3(1 + x^3 + x^6)^2\frac{d}{dx}(1 + x^3 + x^6) = 3(3x^2 + 6x^5)(1 + x^3 + x^6)^2$

EXERCISE 2.5

1. (a) $f'(x) = \frac{2}{(x + 1)^2}$
(b) $f'(x) = \frac{-2x^2 + 2x + 2}{(x^2 + 1)^2}$
(c) $g'(x) = \frac{-x^2 - 1}{(x^2 + 2x - 1)^2}$ (d) $g'(x) = 1$
(e) $y' = \frac{1 - 3x^2}{2\sqrt{x}(x^2 + 1)^2}$
(f) $y' = -\frac{2}{\sqrt{x}(\sqrt{x} - 2)^2}$
(g) $f'(t) = \frac{-2t^2 - 2t + 11}{(t^2 - 3t + 4)^2}$
(h) $g'(t) = \frac{2t^2 - 4t - 4}{(t - 1)^2}$
(i) $f'(x) = \frac{-4x^3 + 2x}{(x^4 - 2x^2 + 1)^2}$
(j) $f'(x) = \frac{ad - bc}{(cx + d)^2}$ (k) $f'(x) = \frac{x^{10} - 60x^5}{(x^5 - 10)^2}$
(l) $f'(x) = \frac{1 + 2x - x^2}{x^2(x + 1)^2}$

2. (a) $\{x|x \neq \frac{1}{2}\}, f'(x) = \frac{5}{(1 - 2x)^2}$
(b) $\{x|x \neq \pm 1\}, f'(x) = \frac{-x^2 - 1}{(x^2 - 1)^2}$
(c) $\{x|x \neq -1, x \neq \frac{3}{2}\}$,
$f'(x) = \frac{1 - 4x}{(x + 1)^2(2x - 3)^2}$
(d) $\{x|x \neq -3, x \neq 1\}$,
$f'(x) = -\frac{2(x^2 + x + 4)}{(x^2 + 2x - 3)^2}$
(e) $\{x|x \neq 1, x \neq -1\}$,
$f'(x) = \frac{-2x^5 - 6x^4 - 2x - 2}{(x^4 - 1)^2}$
(f) $\{x|x \geq 0, x \neq 9\}, f'(x) = \frac{3x\sqrt{x} - 12x}{2(\sqrt{x} - 3)^2}$

3. (a) $x + 2y - 8 = 0$ (b) $9x - y - 13 = 0$
(c) $4x - 25y + 13 = 0$ (d) $x - y - 1 = 0$

4. 7 **6.** $(0,0), (-5,5)$ **7.** $\left(3, \frac{3}{2}\right), \left(-1, \frac{1}{2}\right)$

8. (a) $y' = -\frac{f'(x)}{[f(x)]^2}$ (b) $y' = \frac{xf'(x) - f(x)}{x^2}$
(c) $y' = \frac{f(x) - xf'(x)}{[f(x)]^2}$

EXERCISE 2.6

1. (a) $F'(x) = -21(5 - 3x)^6$
(b) $F'(x) = 80x(2x^2 + 1)^{19}$
(c) $G'(x) = \frac{9x^2 + 6x}{4\sqrt[4]{x^3 + x^2 - 2}}$
(d) $G'(x) = \frac{4x^3 - 1}{2\sqrt{x^4 - x + 1}}$
(e) $y' = \frac{2x + 1}{4(x^2 + x)^{\frac{3}{4}}}$
(f) $y' = -\frac{3(3 + 8x)}{(1 + 3x + 4x^2)^4}$
(g) $y' = -\frac{2(3x^2 + 4x)}{(x^3 + 2x^2 + 1)^3}$
(h) $y' = \frac{4x}{(9 - x^2)^{\frac{3}{2}}}$