10. Find the equations of the tangent lines to the parabola $y = x^2 + x$ that pass through the point $(2, -3)$. Sketch the curve and the tangents.

11. Find the x-coordinates of the points on the hyperbola $xy = 1$ where the tangents from the point $(1, -1)$ intersect the curve.

C **12.** Let

$$f(x) = \begin{cases} 2x + 3 & \text{if } x < -1 \\ x^2 & \text{if } -1 \leqslant x \leqslant 1 \\ 3 - 2x & \text{if } x > 1 \end{cases}$$

(a) Where is f differentiable?

(b) Find an expression for f' and sketch the graphs of f and f'.

13. (a) Sketch the graph of $f(x) = |x^2 - 4|$.

(b) For what values of x is f not differentiable?

(c) Find a formula for f' and sketch its graph.

PROBLEMS PLUS

Suppose that the tangent line at a point P on the curve $y = x^3$ intersects the curve again at a point Q. Show that the slope of the tangent at Q is four times the slope of the tangent at P.

2.4 THE PRODUCT RULE

In this section we develop a formula for the derivative of a product of two functions. It is tempting to guess, as Leibniz did three centuries ago, that the derivative of a product is the product of the derivatives. We can see, however, that this guess is wrong by looking at a particular example. Let

$$f(x) = x \qquad\qquad g(x) = x^2$$

Then $$f'(x) = 1 \qquad\qquad g'(x) = 2x$$

so $$f'(x)g'(x) = 2x$$

But $(fg)(x) = f(x)g(x) = x(x^2) = x^3$, so

$$(fg)'(x) = 3x^2$$

Thus, in general,

$$(fg)' \neq f'g'$$

The correct formula is called the Product Rule and was discovered by Leibniz (soon after his false start).

Product Rule

If both f and g are differentiable, then so is fg and

$$(fg)' = fg' + f'g$$

In Leibniz notation: $\dfrac{d}{dx}[f(x)g(x)] = f(x)\dfrac{d}{dx}g(x) + g(x)\dfrac{d}{dx}f(x)$

In words, the Product Rule says that *the derivative of a product of two functions is the first function times the derivative of the second function plus the second function times the derivative of the first function.*

Proof

Let $F = fg$, that is, $F(x) = (fg)(x) = f(x)g(x)$. Then

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \end{aligned}$$

To evaluate this limit we would like to separate the functions f and g, as in the proof of the Sum Rule. To achieve this separation, we add and subtract the term $f(x + h)g(x)$ in the numerator. This allows us to factor as follows:

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x+h)g(x) + f(x+h)g(x) - f(x)g(x)}{h} \\ &= \lim_{h\to 0} \left[f(x+h)\frac{g(x+h) - g(x)}{h} + g(x)\frac{f(x+h) - f(x)}{h} \right] \\ &= \lim_{h\to 0} f(x+h) \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h\to 0} g(x) \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= f(x)g'(x) + g(x)f'(x) \end{aligned}$$

Notice that

$$\lim_{h\to 0} g(x) = g(x)$$

since $g(x)$ is a constant with respect to h. The reason that

$$\lim_{h\to 0} f(x+h) = f(x)$$

is that f is continuous. (Differentiable functions are continuous. See the Appendix.)

Example 1 Find $\dfrac{dy}{dx}$ if $y = (2x^3 + 5)(3x^2 - x)$.

Solution According to the Product Rule, we have

$$\begin{aligned}\frac{dy}{dx} &= (2x^3 + 5)\frac{d}{dx}(3x^2 - x) + (3x^2 - x)\frac{d}{dx}(2x^3 + 5)\\ &= (2x^3 + 5)(6x - 1) + (3x^2 - x)(6x^2)\end{aligned}$$

If desired, this expression could be simplified as follows:

$$\begin{aligned}\frac{dy}{dx} &= 12x^4 - 2x^3 + 30x - 5 + 18x^4 - 6x^3\\ &= 30x^4 - 8x^3 + 30x - 5\end{aligned}$$

Example 2 Differentiate $f(x) = \sqrt{x}(2 - 3x)$ and simplify.

Solution

$$\begin{aligned}f'(x) &= \sqrt{x}\frac{d}{dx}(2 - 3x) + (2 - 3x)\frac{d}{dx}\sqrt{x}\\ &= \sqrt{x}(-3) + (2 - 3x)\left(\frac{1}{2\sqrt{x}}\right)\\ &= -3\sqrt{x} + \frac{1}{\sqrt{x}} - \frac{3}{2}\sqrt{x}\\ &= \frac{1}{\sqrt{x}} - \frac{9}{2}\sqrt{x}\end{aligned}$$

Notice that we do not actually need the Product Rule to differentiate the functions in Examples 1 and 2. We could have multiplied the factors and proceeded as in Section 2.2. (In fact this is often easier.) But we will later meet functions such as $y = x^2 2^x$ for which the Product Rule must be used.

Example 3 Find the slope of the tangent to the graph of the function $f(x) = (3x^2 + 2)(2x^3 - 1)$ at the point $(1, 5)$.

Solution The Product Rule gives

$$\begin{aligned}f'(x) &= (3x^2 + 2)\frac{d}{dx}(2x^3 - 1) + (2x^3 - 1)\frac{d}{dx}(3x^2 + 2)\\ &= (3x^2 + 2)(6x^2) + (2x^3 - 1)(6x)\end{aligned}$$

There is no need to simplify before substituting $x = 1$.

$$f'(1) = (5)(6) + (1)(6) = 36$$

The slope of the tangent line at $(1, 5)$ is 36.

EXERCISE 2.4

B **1.** Use the Product Rule to find the derivative. Do not simplify your answer.

(a) $f(x) = (2x - 1)(x^2 + 1)$ (b) $f(x) = x(3x - 8)$
(c) $y = x^2(1 + x - 3x^2)$ (d) $y = (x^3 + x^2 + 1)(x^2 + 2)$
(e) $f(t) = (t^4 + t^2 - 1)(t^2 - 2)$ (f) $f(t) = \sqrt[3]{t}(1 - t)$
(g) $F(y) = \sqrt{y}(y - 2\sqrt{y} + 2)$ (h) $G(y) = (y - y^2)(2y - y^{\frac{4}{3}})$

2. Use the Product Rule to differentiate each function. Simplify your answer.

(a) $y = x^3(x^2 + 2x + 3)$ (b) $y = x^{-2}(x^3 - 3x^2 + 6)$
(c) $f(x) = (1 - x^2)(2 - x^3)$ (d) $f(x) = (3x^3 + 4)(1 - 2x^3)$
(e) $f(t) = (6 + t^{-2})(8t^{10} - 5t^3)$ (f) $f(t) = (at + b)(ct^2 - d)$
(g) $g(u) = \sqrt{u}(2 - u^2 + 5u^4)$ (h) $g(v) = (v - \sqrt{v})(v^2 + \sqrt{v})$

3. Find the slope of the tangent to the given curve at the point whose x-coordinate is given.

(a) $y = (1 - 2x)(3x - 4)$, $x = 2$
(b) $y = (1 - x + x^2)(x - 2)$, $x = 1$
(c) $y = x^4(4x^3 + 2)$, $x = -1$
(d) $y = (1 + x - 2x^2)(3x^3 + x - 1)$, $x = 1$
(e) $y = x^{-5}(1 + x^{-1})$, $x = 1$
(f) $y = (2 - 3\sqrt{x})(4 - \sqrt{x})$, $x = 4$

4. If $f(x) = (6x^4 - 3x^2 + 1)(2 - x^3)$, find $f'(1)$ by two methods:

(a) by using the Product Rule;
(b) by expanding $f(x)$ first.

5. Find the equation of the tangent line to the curve $y = (2 - \sqrt{x})(1 + \sqrt{x} + 3x)$ at the point $(1, 5)$.

6. If $f(2) = 3$, $f'(2) = 5$, $g(2) = -1$, and $g'(2) = -4$, find $(fg)'(2)$.

7. If f is a differentiable function, find expressions for the derivatives of the following functions.

(a) $g(x) = xf(x)$ (b) $h(x) = \sqrt{x}f(x)$ (c) $F(x) = x^c f(x)$

8. (a) Use the Product Rule with $g = f$ to show that if f is differentiable, then

$$\frac{d}{dx}[f(x)]^2 = 2f(x)f'(x)$$

(b) Use part (a) to differentiate $y = (2 + 5x - x^3)^2$.

C **9.** (a) Use the Product Rule twice to show that if f, g, and h are differentiable, then

$$(fgh)' = f'gh + fg'h + fgh'$$

(b) Use part (a) to differentiate $y = \sqrt{x}(3x + 5)(6x^2 - 5x + 1)$.

10. (a) Taking $f = g = h$ in Question 9, show that

$$\frac{d}{dx}[f(x)]^3 = 3[f(x)]^2 f'(x)$$

(e) $f'(t) = (t^4 + t^2 - 1)(2t) + (t^2 - 2)(4t^3 + 2t)$

(f) $f'(t) = -\sqrt[3]{t} + \frac{1}{3}t^{-\frac{2}{3}}(1 - t)$

(g) $F'(y) = \sqrt{y}\left[1 - \frac{1}{\sqrt{y}}\right] + (y - 2\sqrt{y} + 2)\frac{1}{2\sqrt{y}}$

(h) $G'(y) = (y - y^2)(2 - \frac{4}{3}y^{\frac{1}{3}}) + (2y - y^{\frac{4}{3}})(1 - 2y)$

2. (a) $y' = 5x^4 + 8x^3 + 9x^2$
(b) $y' = 1 - 12x^{-3}$
(c) $f'(x) = 5x^4 - 3x^2 - 4x$
(d) $f'(x) = -36x^5 - 15x^2$
(e) $f'(t) = 480t^9 + 64t^7 - 90t^2 - 5$
(f) $f'(t) = 3act^2 + 2bct - ad$
(g) $g'(u) = \frac{45}{2}u^{\frac{7}{2}} - \frac{5}{2}u^{\frac{3}{2}} + u^{-\frac{1}{2}}$
(h) $g'(v) = 3v^2 - \frac{5}{2}v^{\frac{3}{2}} + \frac{3}{2}\sqrt{v} - 1$

3. (a) -13 (b) 0 (c) 20 (d) -9 (e) -11 (f) $-\frac{1}{2}$ **4.** 6

5. $x - y + 4 = 0$ **6.** -17

7. (a) $g'(x) = xf'(x) + f(x)$
(b) $h'(x) = \sqrt{x}f'(x) + \frac{1}{2\sqrt{x}}f(x)$
(c) $F'(x) = x^c f'(x) + cx^{c-1}f(x)$

8. (b) $y' = 2(2 + 5x - x^3)(5 - 3x^2)$

9. (b) $y' = \sqrt{x}(3x + 5)(12x - 5) + 3\sqrt{x}(6x^2 - 5x + 1) + \frac{1}{2\sqrt{x}}(3x + 5)(6x^2 - 5x + 1)$

10. (b) $y' = 3(1 + x^3 + x^6)^2\frac{d}{dx}(1 + x^3 + x^6) = 3(3x^2 + 6x^5)(1 + x^3 + x^6)^2$

EXERCISE 2.5

1. (a) $f'(x) = \frac{2}{(x + 1)^2}$
(b) $f'(x) = \frac{-2x^2 + 2x + 2}{(x^2 + 1)^2}$
(c) $g'(x) = \frac{-x^2 - 1}{(x^2 + 2x - 1)^2}$ (d) $g'(x) = 1$
(e) $y' = \frac{1 - 3x^2}{2\sqrt{x}(x^2 + 1)^2}$
(f) $y' = -\frac{2}{\sqrt{x}(\sqrt{x} - 2)^2}$
(g) $f'(t) = \frac{-2t^2 - 2t + 11}{(t^2 - 3t + 4)^2}$
(h) $g'(t) = \frac{2t^2 - 4t - 4}{(t - 1)^2}$
(i) $f'(x) = \frac{-4x^3 + 2x}{(x^4 - 2x^2 + 1)^2}$
(j) $f'(x) = \frac{ad - bc}{(cx + d)^2}$ (k) $f'(x) = \frac{x^{10} - 60x^5}{(x^5 - 10)^2}$
(l) $f'(x) = \frac{1 + 2x - x^2}{x^2(x + 1)^2}$

2. (a) $\{x|x \neq \frac{1}{2}\}$, $f'(x) = \frac{5}{(1 - 2x)^2}$
(b) $\{x|x \neq \pm 1\}$, $f'(x) = \frac{-x^2 - 1}{(x^2 - 1)^2}$
(c) $\{x|x \neq -1, x \neq \frac{3}{2}\}$, $f'(x) = \frac{1 - 4x}{(x + 1)^2(2x - 3)^2}$
(d) $\{x|x \neq -3, x \neq 1\}$, $f'(x) = -\frac{2(x^2 + x + 4)}{(x^2 + 2x - 3)^2}$
(e) $\{x|x \neq 1, x \neq -1\}$, $f'(x) = \frac{-2x^5 - 6x^4 - 2x - 2}{(x^4 - 1)^2}$
(f) $\{x|x \geq 0, x \neq 9\}$, $f'(x) = \frac{3x\sqrt{x} - 12x}{2(\sqrt{x} - 3)^2}$

3. (a) $x + 2y - 8 = 0$ (b) $9x - y - 13 = 0$
(c) $4x - 25y + 13 = 0$ (d) $x - y - 1 = 0$

4. 7 **6.** $(0, 0)$, $(-5, 5)$ **7.** $\left(3, \frac{3}{2}\right)$, $\left(-1, \frac{1}{2}\right)$

8. (a) $y' = -\frac{f'(x)}{[f(x)]^2}$ (b) $y' = \frac{xf'(x) - f(x)}{x^2}$
(c) $y' = \frac{f(x) - xf'(x)}{[f(x)]^2}$

EXERCISE 2.6

1. (a) $F'(x) = -21(5 - 3x)^6$
(b) $F'(x) = 80x(2x^2 + 1)^{19}$
(c) $G'(x) = \frac{9x^2 + 6x}{4\sqrt[4]{x^3 + x^2 - 2}}$
(d) $G'(x) = \frac{4x^3 - 1}{2\sqrt{x^4 - x + 1}}$
(e) $y' = \frac{2x + 1}{4(x^2 + x)^{\frac{3}{4}}}$
(f) $y' = -\frac{3(3 + 8x)}{(1 + 3x + 4x^2)^4}$
(g) $y' = -\frac{2(3x^2 + 4x)}{(x^3 + 2x^2 + 1)^3}$
(h) $y' = \frac{4x}{(9 - x^2)^{\frac{3}{2}}}$