Solving Quadratic Trigonometric Equations

6.6 SKILL BUILDER

Often in mathematics, it is necessary to combine skills and strategies you have used before to deal with more complicated problems. A **quadratic trigonometric equation** is a trigonometric equation with $\sin x$, $\cos x$, or $\tan x$, and has 2 as its highest power. For instance, $2\sin^2 x + \sin x = 1$ is a quadratic trigonometric equation. These equations are new to you, but you can solve them by combining methods for solving quadratic equations and linear trigonometric equations.

Solving by Factoring

Recall that factoring can help solve some quadratic equations.

Example 1

Solve $x^2 - 2x = 15$.

Solution

$x^2 - 2x = 15$	Express the equation in the form $ax^2 + bx + c = 0$.
$x^2 - 2x - 15 = 0$	Factor.
$(x - 5)(x + 3) = 0$	Set each factor equal to 0 and solve.
$x - 5 = 0$ and $x + 3 = 0$	
$x = 5$ and $x = -3$	

This strategy can be applied to quadratic trigonometric equations.

Factoring to Solve Quadratic Trigonometric Equations

Factoring can be used to solve some quadratic trigonometric equations once the equation has been expressed in standard form.

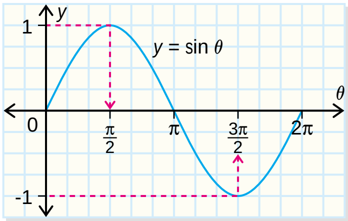
Example 2

Solve $\sin^2 x = 1$, $0 \le x \le 2\pi$.

Solution

$\sin^2 x = 1$	Rearrange the equation so one side equals 0.
$\sin^2 x - 1 = 0$	Factor.
$(\sin x - 1)(\sin x + 1) = 0$	Set each factor equal to 0 and solve.
$\sin x - 1 = 0$ and $\sin x + 1 = 0$	
$\sin x = 1$ and $\sin x = -1$	

In the graph of $y = \sin x$, notice that $y = 1$ when $x = \frac{\pi}{2}$, and $y = -1$ when $x = \frac{3\pi}{2}$.



$$\sin \frac{\pi}{2} = 1 \quad \text{and} \quad \sin \frac{3\pi}{2} = -1$$

Therefore, $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Using Special Triangles

Sometimes the special triangles can be used.

Example 3

Solve $2\cos^2 2\theta - 1 = -\cos 2\theta$, $0 \le \theta \le 2\pi$.

Solution

$$2\cos^2 2\theta - 1 = -\cos 2\theta \quad \text{Rearrange so one side equals 0.}$$

$$2\cos^2 2\theta + \cos 2\theta - 1 = 0 \quad \text{Factor.}$$

$$(2\cos 2\theta - 1)(\cos 2\theta + 1) = 0 \quad \text{Set each factor equal to 0 and solve.}$$

$$2\cos 2\theta - 1 = 0 \quad \text{and} \quad \cos 2\theta + 1 = 0$$

$$2\cos 2\theta = 1 \quad \text{and} \quad \cos 2\theta = -1$$

$$\cos 2\theta = \frac{1}{2}$$

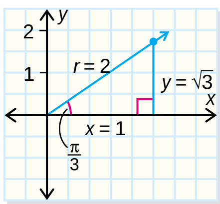
The equation $\cos 2\theta = \frac{1}{2}$ can be solved using the $\frac{\pi}{3} - \frac{\pi}{6}$ special triangle. Cosine is positive in quadrants I and IV. In this case, $\frac{\pi}{3}$ is the related angle.

In quadrant I: $x = 1$, $r = 2$

$$\therefore 2\theta = \frac{\pi}{3}$$

$$\theta = \frac{\pi}{(3)(2)}$$

$$\theta = \frac{\pi}{6}$$

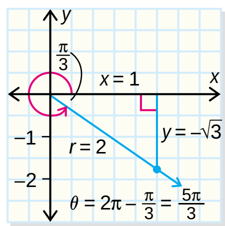


In quadrant IV: $x = 1$, $r = 2$

$$\therefore 2\theta = \frac{5\pi}{3}$$

$$\theta = \frac{5\pi}{(3)(2)}$$

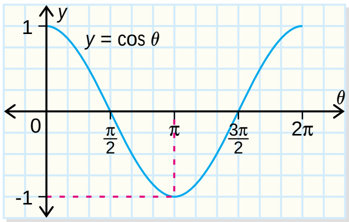
$$\theta = \frac{5\pi}{6}$$



However, the period of the equation is π. Add π to the solutions to find all the solutions in the domain.

$$\theta_1 = \frac{\pi}{6} \qquad \theta_2 = \frac{\pi}{6} + \pi = \frac{7\pi}{6} \qquad \theta_3 = \frac{5\pi}{6} \qquad \theta_4 = \frac{5\pi}{6} + \pi = \frac{11\pi}{6}$$

Use the graph of $y = \cos\theta$ to solve $2\theta = -1$.



$$2\theta = \pi$$
$$\theta = \frac{\pi}{2}$$

However, the period of the equation is π. Add π to the solutions to find all the solutions in the domain.

$$\theta_5 = \frac{\pi}{2} \qquad \theta_6 = \frac{\pi}{2} + \pi$$
$$= 3\frac{\pi}{2}$$

Therefore, $2\cos^2\theta - 1 = -\cos\theta$, $0 \le \theta \le 2\pi$, has six solutions.

$$\theta_1 = \frac{\pi}{6} \qquad \theta_2 = \frac{\pi}{2} \qquad \theta_3 = \frac{5\pi}{6} \qquad \theta_4 = \frac{7\pi}{6} \qquad \theta_5 = \frac{3\pi}{2} \qquad \theta_6 = \frac{11\pi}{6}$$

Using a Calculator

Often you must use the Pythagorean identity and a calculator to solve an equation.

Example 4

Solve $8 + 13\sin x = 12\cos^2 x$, $0° \le x \le 360°$.

Solution

$8 + 13\sin x = 12\cos^2 x$	Rearrange so one side equals 0.
$-12\cos^2 x + 13\sin x + 8 = 0$	Express the equation in terms of $\sin x$ ($\cos^2 x = 1 - \sin^2 x$).
$-12(1 - \sin^2 x) + 13\sin x + 8 = 0$	Expand.
$-12 + 12\sin^2 x + 13\sin x + 8 = 0$	Simplify.
$12\sin^2 x + 13\sin x - 4 = 0$	Factor.
$(3\sin x + 4)(4\sin x - 1) = 0$	Set each factor equal to 0 and solve.

$$3\sin x + 4 = 0 \quad \text{and} \quad 4\sin x - 1 = 0$$
$$3\sin x = -4 \quad \text{and} \quad 4\sin x = 1$$
$$\sin x = -\frac{4}{3} \quad \text{and} \quad \sin x = \frac{1}{4}$$

The equation $\sin x = -\frac{4}{3}$ has no solutions, since $-1 \le \sin x \le 1$.

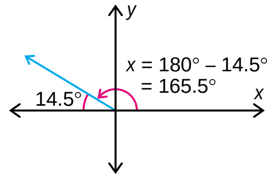
The equation $\sin x = \frac{1}{4}$ has two solutions, and x is an angle in quadrants I or II. Use a scientific or graphing calculator.

$$x = \sin^{-1}\left(\frac{1}{4}\right)$$
$$x \doteq 14.5°$$

The angle in quadrant II has 14.5° as its related angle.

Therefore, $x \doteq 180° - 14.5°$ or 165.5°.

The equation $8 + 13 \sin x = 12 \cos^2 x$, $0° \le x \le 360°$ has two solutions, $x \doteq 14.5°$ and $x \doteq 165.5°$.



Using Graphing Technology

Graphing technology can be used to solve a quadratic trigonometric equation. This approach will work even when the equation cannot be factored.

Example 5

Solve $6 \sin^2 x + \sin x - 2 = 0$, $0 \le x \le 2\pi$, using graphing technology.

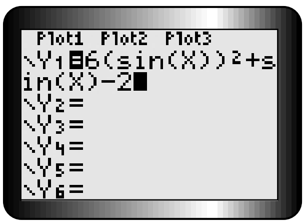
Solution

Find the solutions to this equation by graphing $y = 6 \sin^2 x + \sin x - 2$ over the domain $0 \le x \le 2\pi$ and locating the zeros, or x-intercepts.

1. Put the calculator in radian mode.

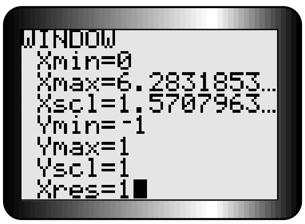
   On the TI-83 Plus, press [MODE]. Then scroll down to Radian and press [ENTER].

2. Enter the relation into the equation editor.



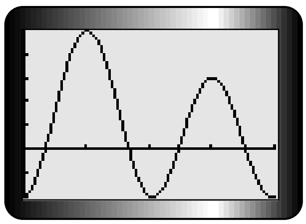
3. Set the window for the given domain.

   Xmin = 0, Xmax = 2π, and Xscl = $\pi/2$.



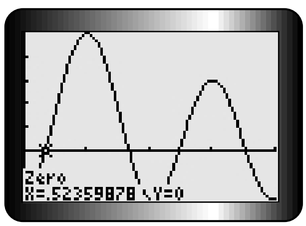
4. Graph by using zoom fit.

   Press [ZOOM] [0].



5. Determine all the zeros.

   Press [2nd] [TRACE] [2], then choose the left bound and right bound. Press [ENTER] on Guess.



6. The three other zeros are found similarly.

The equation $6\sin^2 x + \sin x - 2 = 0$ has solutions $x = 0.5236$, $x = 2.6180$, $x = 3.8713$, and $x = 5.5535$, $0 \le x \le 2\pi$.

Key Ideas

- Any trigonometric equation with $\sin x$, $\cos x$, or $\tan x$, with 2 as its highest power, is called a **quadratic trigonometric equation**. For example, $2\sin^2 x + \sin x = 1$ is a quadratic trigonometric equation.
- A quadratic trigonometric equation must be expressed in terms of the same trigonometric ratio before it can be solved.
- Factoring often helps when solving a quadratic expression. Once the expression has been factored, set each factor equal to 0 to obtain two linear trigonometric equations that can be solved.
- It is possible that one of the factors may not lead to a solution of the original equation. This occurs when $\sin\theta > 1$, $\cos\theta > 1$, $\sin\theta < -1$, or $\cos\theta < -1$.
- The period and the domain of the corresponding functions must be considered when determining all possible solutions.
- A close approximation to the solution for any quadratic trigonometric equation can be found by graphing the corresponding trigonometric function using graphing technology and determining the zeros of the function over the given domain.

Practise, Apply, Solve 6.6

A

1. Factor.

(a) $5x^2 - 10x$

(b) $x^2 + 13x + 40$

(c) $10x^2 - 11x - 6$

(d) $x^2 - 81$

2. Factor.

(a) $\sin^2\theta - \sin\theta$

(b) $\cos^2\theta - 2\cos\theta + 1$

(c) $3\sin^2\theta - \sin\theta - 2$

(d) $4\cos^2\theta - 1$

3. Verify that the value of θ is a solution of each equation.

(a) $2\sin^2\theta = 1$, $\theta = \frac{3\pi}{4}$

(b) $4\cos^2 2\theta = 3$, $\theta = \frac{\pi}{12}$

(c) $2\sin^2\theta - \sin\theta = 1$, $\theta = \frac{\pi}{2}$

4. Solve each equation for x, $0° \leq x \leq 360°$.

(a) $\sin x \cos x = 0$

(b) $\sin x (\cos x - 1) = 0$

(c) $(\sin x + 1) \cos x = 0$

(d) $\cos x (2 \sin x - \sqrt{3}) = 0$

(e) $(\sqrt{2} \sin x - 1)(\sqrt{2} \sin x + 1) = 0$

(f) $(\sin x - 1)(\cos x + 1) = 0$

5. Solve each equation for x, $0 \leq x \leq 2\pi$.

(a) $(2 \sin x - 1) \cos x = 0$

(b) $(\sin x + 1)^2 = 0$

(c) $(2 \cos x + \sqrt{3}) \sin x = 0$

(d) $(2 \cos x - 1)(2 \sin x + \sqrt{3}) = 0$

(e) $(\sqrt{2} \cos x - 1)(\sqrt{2} \cos x + 1) = 0$

(f) $(\sin x + 1)(\cos x - 1) = 0$

B

6. Solve for θ to the nearest degree, $0° \leq \theta \leq 360°$.

(a) $\sin^2 \theta = 1$ (b) $\cos^2 \theta = 1$ (c) $\tan^2 \theta = 1$

(d) $4 \cos^2 \theta = 1$ (e) $3 \tan^2 \theta = 1$ (f) $2 \sin^2 \theta = 1$

7. (a) Write $2 \sin^2 x - \sin x - 1$ in factored form.

(b) Use the factors from (a) to solve $2 \sin^2 x - \sin x - 1 = 0$, $0 \leq x \leq 2\pi$.

8. (a) Write $2 \cos^2 x + \cos x - 1$ in factored form.

(b) Use the factors in (a) to solve $2 \cos^2 x + \cos x - 1 = 0$, $0° \leq x \leq 360°$.

9. Solve for x to the nearest degree, $0° \leq x \leq 360°$.

(a) $2 \sin^2 x - \sin x = 0$ (b) $\cos^2 x = \cos x$

(c) $2 \tan^2 x + \tan x - 3 = 0$ (d) $6 \sin^2 x - \sin x = 1$

(e) $\cos^2 x - 6 \cos x + 5 = 0$ (f) $4 \sin^2 x - 3 = -\sin \mathrm{x}$

10. Solve for θ to the nearest hundredth of a radian, $0 \leq \theta \leq 2\pi$.

(a) $2 \cos^2 \theta + \cos \theta - 1 = 0$ (b) $2 \sin^2 \theta = 1 - \sin \theta$

(c) $\cos^2 \theta = 2 + \cos \theta$ (d) $2 \sin^2 \theta + 5 \sin \theta - 3 = 0$

(e) $3 \tan^2 \theta - 2 \tan \theta = 1$ (f) $12 \sin^2 \theta + \sin \theta - 6 = 0$

11. Solve for x to the nearest degree, $0° \leq x \leq 360°$.

(a) $\cos^2 2x = -\cos 2x$ (b) $2 \sin^2 \left(\frac{x}{2}\right) = 1$

12. Solve for θ to the nearest hundredth of a radian, $0 \leq \theta \leq 2\pi$.

(a) $\cos^2 \theta - \sin^2 \theta = 1$ (b) $\sin \theta - \cos^2 \theta - 1 = 0$

(c) $2 \sin - \cos^2 \theta = 2$ (d) $13 - 15 \sin^2 \theta + \cos \theta = 0$

13. Solve for θ, using graphing technology. Answer to the nearest degree, $0° \leq \theta \leq 360°$.

(a) $3 \sin^2 \theta + 2 \sin \theta = 0$

(b) $8 \cos^2 \theta - 2 \cos \theta - 1 = 0$

(c) $4 \cos^2 \left(\frac{\theta}{2}\right) - 3 \cos \left(\frac{\theta}{2}\right) = 0$

(d) $5 \sin^2 (2\theta) + 14 \sin (2\theta) = 3$

14. Solve for x using graphing technology. Answer to the nearest hundredth, $0 \leq x \leq 2\pi$.

(a) $3 \cos^2 x = 5 \cos x$

(b) $2 \sin^2 x + 3 \sin x - 2 = 0$

(c) $4 \cos^2 x = 7 \cos x + 2$

(d) $3 \sin^2 x - 2 \sin x - 1 = 0$

15. **Check Your Understanding**

(a) Give an example of a quadratic trigonometric equation.

(b) Can factoring be used to solve all quadratic trigonometric equations? Explain.

(c) What two factors play a role when determining the number of solutions of a quadratic trigonometric equation?

C

16. Solve each equation without using graphing technology.

(a) $3 \tan^2 (2x) = 1, 0° \leq x \leq 360°$

(b) $\sqrt{2} \sin \theta = \sqrt{3} - \cos \theta, 0 \leq \theta \leq 2\pi$. (Hint: Square both sides of the equation and check for extraneous roots.)

The Chapter Problem—What Time Is It?

Apply what you have learned to answer these questions about the Chapter Problem on page 494.

CP7. A 70-cm long pendulum is released at a distance of 8 cm from rest.

(a) Determine the period and the time of one complete swing.

(b) Graph the function that models this situation. Explain the meaning of positive and negative values of the dependent variable, x.

(c) Determine the pendulum's distance from the rest position after 2 s.

(d) Determine four possible values for t when the pendulum is 6 cm from the centre point.

CP8. Use the model $x = M \cos \left(t \sqrt{\frac{980}{l}}\right)$ to determine an expression that represents the period of the function when $l = 70$ cm.

CP9. How long does the pendulum in a grandfather clock take to complete one full period? (A period is two complete swings.)

CP10. Use your answers from CP8 and CP9 to create an equation to determine the length of the pendulum in a grandfather clock. Solve the equation.