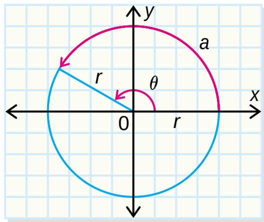
5.4 Radian Measure

So far, you have measured angles in degrees, with 360° being one revolution around a circle. There is another way to measure angles called **radian measure**. With radian measure, the arc length of a circle is compared to the radius of the circle in the ratio $\frac{\text{arc length}}{\text{radius}}$. A radian does not have a specific unit but is a real number. Radian measure has many real-life applications for periodic functions.

In radian measure,

$$\theta = \frac{\text{arc length}}{\text{radius}}$$

$$\theta = \frac{a}{r}$$



What is the radian measure for one complete revolution around a circle, that is, when $\theta = 360°$? Once around the circle is the circumference of the circle. For a complete circle, the arc length is equal to the circumference. If the radius is r, then $C = 2\pi r$.

If $\theta = \frac{a}{r}$, then

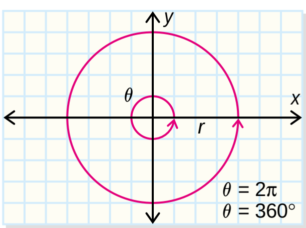
$$\theta = \frac{2\pi r}{r}$$

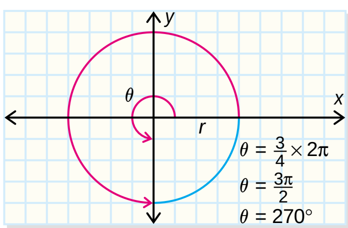
$$\theta = 2\pi$$

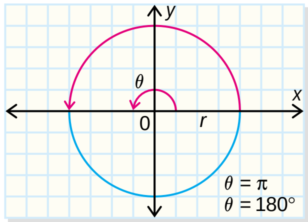
Notice that no units are attached to this value because it is a real number. The value 2π can be converted to an approximate real number. For instance, if two decimal places are required, then replace π with 3.14. Then,

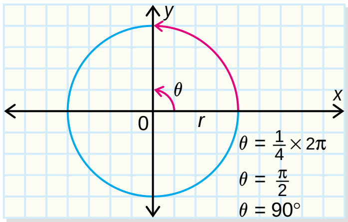
$$\begin{aligned}\theta &= 2\pi \\ &\doteq 2(3.14) \\ &= 6.28\end{aligned}$$

Observe in these diagrams how degree measure and radian measure describe the same angle θ.









In trigonometry, it is often necessary to convert from degree measure to radian measure, and vice versa. When converting, the proportion $\frac{\text{radians}}{\pi} = \frac{\text{degrees}}{180°}$ is often useful. When using the proportion, substitute any three known values to determine the fourth value.

There are two values π can have. If the answer is in degrees, $\pi = 180°$. If it is in radians, $\pi \doteq 3.14$ radians.

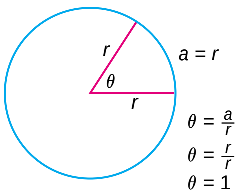
Example 1

What does one radian look like and how many degrees is it equivalent to?

Solution

One radian occurs when the arc length and the radius are the same length. Use the proportion relating radians and degrees. Note that $\pi \doteq 3.14$ radians.

$$\begin{aligned} \frac{\text{radians}}{\pi} &= \frac{\text{degrees}}{180°} \\ \frac{1}{\pi} &= \frac{x}{180°} \\ x &= \frac{180°}{\pi} \\ &\doteq 57.3° \end{aligned}$$



Therefore, one radian is about 57°.

Example 2

Convert 120° to radians and round to two decimals.

Solution

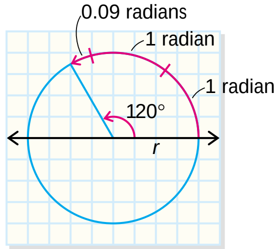
Use the proportion relating radians and degrees.

$$\begin{aligned} \frac{\text{radians}}{\pi} &= \frac{\text{degrees}}{180°} \\ \frac{x}{\pi} &= \frac{120°}{180°} \\ x &= \frac{120°\pi}{180°} \\ x &= \frac{2\pi}{3} \end{aligned}$$

Notice that the number has no units and is now a real number. Now find the approximate value using $\pi \doteq 3.14$.

$$x = \frac{2(3.14)}{3}$$
$$\doteq 2.09$$

Check visually.



Example 3

Change $\frac{5\pi}{3}$ to degree measure.

Solution

One method is to use the proportion relating radians and degrees.

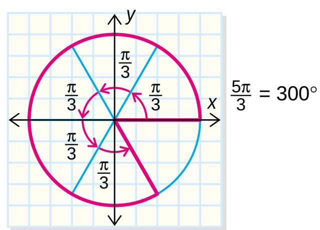
$$\frac{\text{radians}}{\pi} = \frac{\text{degrees}}{180°}$$
$$\frac{\frac{5\pi}{3}}{\pi} = \frac{x}{180°}$$
$$\frac{\frac{5}{3}}{1} = \frac{x}{180°}$$
$$x = \frac{5}{3}(180°)$$
$$x = 300°$$

Then $\frac{5\pi}{3} = 300°$.

Another method is to simply substitute $\pi = 180°$.

$$\frac{5\pi}{3} = \frac{5(180°)}{3}$$
$$\frac{5\pi}{3} = 300°$$

Check visually.



Example 4

The motion of a certain pendulum is modelled by $d = \cos\left(\frac{\sqrt{9.8}}{2}t\right)$, where d is the distance in metres of the arc length from the release point and t is the time in seconds since release.

(a) Make a table in 1-s increments for $0 \leq t \leq 10$. Round distances to the nearest metre.

(b) Use the table to draw the graph.

(c) What is the length of one cycle? Explain why the graph is periodic in the context of the question.

(d) What is the maximum displacement from rest?

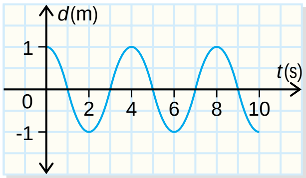
(e) State the amplitude of the function.

Solution

(a) Evaluate $d = \cos\left(\frac{\sqrt{9.8}}{2}t\right)$, for $0 \le t \le 10$, and complete the table.

t (s)	0	1	2	3	4	5	6	7	8	9	10
d (m)	1	0	−1	0	1	0	−1	0	1	0	−1

(b)

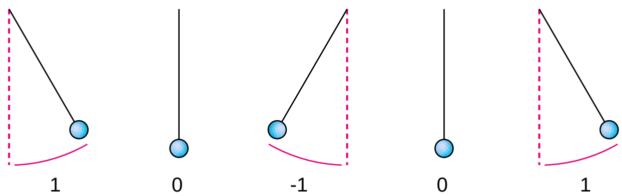


(c) One cycle is 4 s.

The graph is periodic because it repeats itself every 4 s. The pendulum swings 1 m away from rest, through rest to a point 1 m on the other side of rest, and then returns, passing through rest to its original position.

(d) The maximum displacement is 1 m.

(e) The amplitude is 1.



Consolidate Your Understanding

1. Show how degree measure and radian measure are related.

2. Explain the difference between $\pi = 180°$ and $\pi = 3.14$.

Focus 5.4

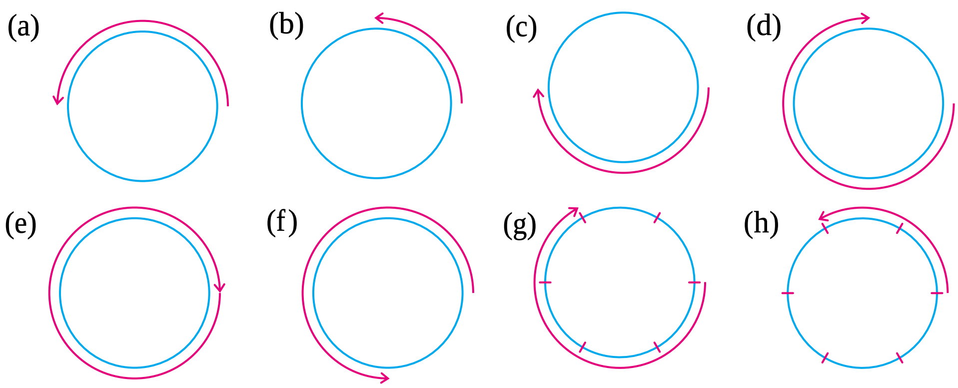
Key Ideas

- Angles can be measured using degrees or radians.
- A radian has no specified unit. It is simply a real number.
- π radians $= 180°$ or π radians $\doteq 3.14$ radians.
- The proportion $\frac{\text{radians}}{\pi} = \frac{\text{degrees}}{180°}$ can be used to convert between radian and degree measures.
- Radian measure has many practical applications to periodic phenomena.

Practise, Apply, Solve 5.4

A

1. A point is rotated about a circle of radius 1. Its start and finish are shown. State the rotation in radian measure and in degree measure.



2. Sketch each rotation about a circle of radius 1.

(a) π (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{4\pi}{3}$

(e) $\frac{5\pi}{3}$ (f) $-\pi$ (g) $-\frac{\pi}{2}$ (h) $-\frac{\pi}{4}$

3. Convert to degree measure.

(a) $\frac{2\pi}{3}$ (b) $-\frac{5\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $-\frac{3\pi}{4}$

(e) $\frac{7\pi}{6}$ (f) $-\frac{3\pi}{2}$ (g) $\frac{11\pi}{6}$ (h) $-\frac{9\pi}{2}$

4. Sketch the approximate location of each radian measure on a unit circle. Do not convert to degree measure.

(a) 3.14 (b) 2 (c) 1.5 (d) 4.2 (e) -5.3

5. **Knowledge and Understanding:** Convert to radian measure.

(a) 90° (b) 270° (c) $-180°$ (d) 45°

(e) $-135°$ (f) 60° (g) 240° (h) $-120°$

6. Sketch each angle in standard position and state its related acute angle.

(a) $\frac{\pi}{4}$ (b) $\frac{2\pi}{3}$ (c) $-\frac{\pi}{6}$

(d) $-\frac{3\pi}{2}$ (e) $\frac{5\pi}{4}$ (f) $\frac{5\pi}{3}$

B

7. (a) Graph $y = \sin \theta$ for $-2\pi \leq \theta \leq 2\pi$. Use a table with $\frac{\pi}{6}$ increments.

(b) What are the coordinates of all maximum and minimum points for this domain?

(c) State the location of all zeros of the function for this domain.

8. (a) Graph $y = \cos \theta$ for $-2\pi \leq \theta \leq 2\pi$. Use a table with $\frac{\pi}{4}$ increments.

(b) What are the coordinates of all maximum and minimum points for this domain?

(c) State the location of all zeros of the function for this domain.

9. (a) Graph $y = \tan \theta$ for $-2\pi \leq \theta \leq 2\pi$. Use a table with $\frac{\pi}{6}$ increments.

(b) State the equation of all asymptotes within this domain.

(c) State the location of all zeros of the function for this domain.

10. Sketch the graph within the given domain.

(a) $y = \sin \theta$, $-2\pi \leq \theta \leq 2\pi$ (b) $y = \sin \theta$, $-180° \leq \theta \leq 540°$

11. Sketch the graph within the given domain.

(a) $y = \cos \theta$, $0° \leq \theta \leq 360°$ (b) $y = \cos \theta$, $-\pi \leq \theta \leq 3\pi$

12. Sketch the graph within the given domain.

(a) $y = \tan \theta$, $-180° \leq \theta \leq 180°$ (b) $y = \tan \theta$, $-\pi \leq \theta \leq \frac{3\pi}{2}$

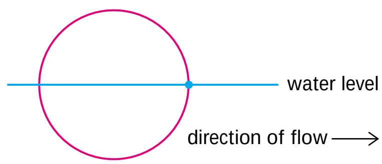
13. Determine all values of θ, to the nearest degree, for $-360° \leq \theta \leq 360°$.

(a) $\sin \theta = -\frac{1}{2}$ (b) $\cos \theta = 0.825$ (c) $\tan \theta = 13.623$

14. Determine all values of θ, to one decimal place, for $-\pi \leq \theta \leq 3\pi$.

(a) $\sin \theta = \frac{7}{8}$ (b) $\cos \theta = -0.5$ (c) $\tan \theta = -4.25$

15. A water wheel of radius 1 m sits in a stream as shown.



(a) Draw, for one complete revolution of the wheel, a sequence of right angle triangles to represent the position of a point on the water wheel for every rotation of $\frac{\pi}{6}$.

(b) Make a table with intervals of $\frac{\pi}{6}$ to show the displacement from the surface of the stream of the indicated point as it rotates from 0 to 2π.

(c) Use the table to graph displacement from surface versus angle of rotation.

(d) Describe the graph and write an equation that models the situation.

16. **Application:** A buoy rises and falls as it rides the waves. The equation $h(t) = \cos \frac{\pi}{5}t$ models the displacement of the buoy in metres at t seconds.

(a) Graph the displacement from 0 to 20 s, in 2.5-s intervals.

(b) Determine the period of the function from the graph. Determine the period algebraically from the equation.

(c) What is the displacement at 35 s?

(d) At what time, to the nearest second, does the displacement first reach -0.8 m?

17. **Communication:** A spring bounces up and down according to the model $d(t) = 0.5 \cos 2t$, where d is the displacement in centimetres from the rest position and t is the time in seconds. The model does not consider the effects of gravity.

(a) Make a table for $0 \le t \le 9$, using 0.5-s intervals.

(b) Draw the graph.

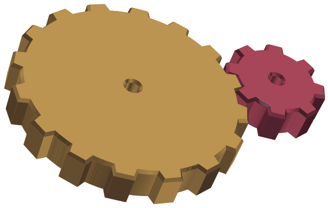
(c) Explain why the function models periodic behaviour.

(d) What is the relationship between the amplitude of the function and the displacement of the spring from its rest position?

18. **Check Your Understanding:** Explain how periodic phenomena can be measured in degrees or real numbers. Give an example to support your answer.

C

19. **Thinking, Inquiry, Problem Solving:** A gear of radius 1 m turns in a counterclockwise direction and drives a larger gear of radius 3 m. Both gears have their central axis along the horizontal.



(a) Which direction is the larger gear turning?

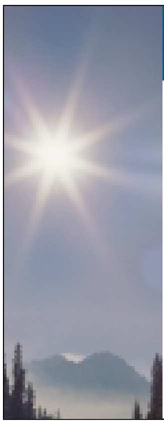
(b) If the period of the smaller gear is 2 s, what is the period of the larger gear?

(c) Make a table in convenient intervals for each gear, to show the vertical displacement, d, of the point where the two gears first touched. Begin the table at 0 s and end it at 12 s. Graph vertical displacement versus time.

(d) What is the displacement of the point on the large wheel when the drive wheel first has a displacement of -0.5 m?

(e) What is the displacement of the drive when the large wheel first has a displacement of 2 m?

(f) What is the displacement of the point on the large wheel at 5 min?



The Chapter Problem—How Much Daylight?

CP8. Revisit your graph of the data for the two-year period in the table.

(a) Evaluate $f(3)$.

(b) Show that $f(3) = f(3 + 12)$ and explain why this should be true.

(c) Explain why there are other values that are about the same as $f(3)$.

CP9. (a) Approximate the average number of daylight hours for $f(30)$.

(b) Extend the graph to evaluate $f(-10)$.

(c) Which months correspond to $f(30)$ and $f(-10)$?