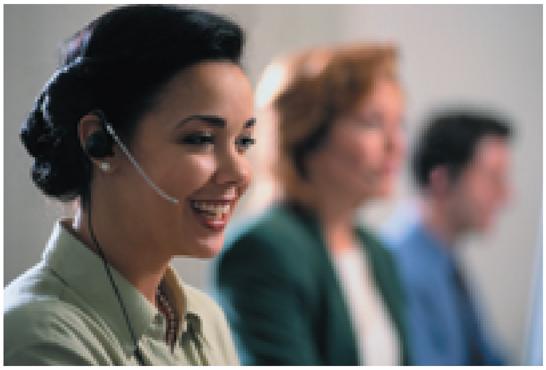
Geometric Series 2.3

A large telemarketing call centre will be closed on Monday due to an ice storm, and the employees are notified on Sunday. The company has already set up an emergency phone "tree." The company's president calls three employees. Then each of these three employees calls three other people, and so on.



Think, Do, Discuss

1. Start with the company's president at the top, and draw a diagram of the phone tree for the first four rounds of calls. The diagram represents the sequence of the number of employees notified at each round.
2. How many employees were notified by the president, who made the first round of calls? How many employees were notified during the second round of calls? the third round of calls? the fourth round of calls?
3. The number of employees notified during each round of calls forms a sequence. What do you call this sequence? Explain. Determine the general term, t_n, to represent the number of employees notified during the nth round.
4. Write the sequence that represents the number of employees notified for the first seven rounds of calls. Find the total number of employees notified after the first seven rounds of calls.
5. The sum of the terms of a geometric sequence is a **geometric series**. The sum of the sequence in step 4 is S_7, where $S_7 = t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7$. Write the series, substituting the appropriate values for t_1 to t_7.
6. To develop a formula for the sum of the geometric series, begin by multiplying both sides of the equation in step 5 by the common ratio, $r = 3$. Write the terms in $3S_7$ so that t_1 of $3S_7$ is below t_2 of S_7 and t_2 of $3S_7$ is below t_3 of S_7, and so on. Compare S_7 to $3S_7$. What is the same? What is different? Would there be so many common terms if you had multiplied S_7 by a number other than the common ratio of $r = 3$? Explain.
7. Subtract S_7 from $3S_7$. What values remain on the right side? Which terms of the geometric sequence do these values represent?
8. $2S_7$ is now the sum of only two terms. What must you do to both sides so that the left side is S_7? Find S_7. What does this sum represent?
9. Use the method in steps 7 and 8 to determine the total number of employees notified after ten rounds of calls.

10. The general term of a geometric sequence is $t_n = 3(4)^{n-1}$. Use the method in steps 5 to 8 to find S_8, the geometric series, or sum, of the first eight terms.

11. A geometric sequence has the general term $t_n = ar^{n-1}$, and a and r are known. Suggest a formula for finding S_n, the geometric series of this sequence.

12. List the first five terms of the series if the first term is a and the common ratio, r, is 1. What is the sum of these five terms, S_5?

Focus 2.3

Key Ideas

- The general term of a geometric sequence is $t_n = ar^{n-1}$, where a is the first term of the sequence and r is the common ratio.
- The sum of the terms of a geometric sequence is a **geometric series**. The sum is written

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$$

- The sum of the first n terms of a geometric series can also be written

$$S_n = \frac{t_{n+1} - t_1}{r - 1},\ r \neq 1$$

- In any geometric sequence, $t_{n+1} = r(t_n) = r(ar^{n-1}) = ar^{1+n-1} = ar^n$, and $t_1 = a$. Substituting these values in $\frac{t_{n+1} - t_1}{r - 1}$ gives

$$S_n = \frac{ar^n - a}{r - 1}$$
$$= \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

Example 1

Find S_8, the sum of the first eight terms of each series.

(a) $2 - 6 + 18 - 54 + \ldots$ (b) $200 + 100 + 50 + 25 + \ldots$

Solution

(a) The series is geometric, and $a = 2$ and $r = -3$.

Method 1: Use $S_n = \frac{t_{n+1} - t_1}{r - 1}$.

To find S_8, first find $t_{n+1} = ar^n$.

$t_9 = ar^8 = 2(-3)^8 = 13\ 122$

In this case, $t_1 = 2$ and $r = -3$.

$$S_8 = \frac{13\ 122 - 2}{-3 - 1}$$
$$= \frac{13\ 120}{-4}$$
$$= -3280$$

Method 2: Use $S_n = \frac{a(r^n - 1)}{r - 1}$.

In this case, $a = 2$, $r = -3$, and $n = 8$.

$$S_8 = \frac{2[(-3)^8 - 1]}{-3 - 1}$$
$$= \frac{2(6561 - 1)}{-3 - 1}$$
$$= \frac{13\ 120}{-4}$$
$$= -3280$$

(b) The series is geometric, and $a = 200$, $r = \frac{1}{2}$, and $n = 8$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$ Substitute the values of a, r, and n.

$$S_8 = \frac{200\left[\left(\frac{1}{2}\right)^8 - 1\right]}{\frac{1}{2} - 1}$$ Simplify.

$$\doteq \frac{200(-0.996\ 093\ 75)}{-\frac{1}{2}}$$

$$= 398.4375$$

(**Hint:** To get the fractional equivalent of a decimal using the TI-83 Plus calculator, press MATH 1 ENTER.)

$$= \frac{6375}{16}$$

Example 2

A new lottery offers to pay the grand prize winner in one of two ways:

Option A: \$10 000 000 now

Option B: A payment each day for 30 days: \$0.01 on the first day, \$0.02 on the second day, \$0.04 on the third day, \$0.08 on the fourth day, and so on

For the grand prize winner, which option results in the biggest grand prize?

Solution

Option B can be represented by the following series:

$$S_{30} = 0.01 + 0.02 + 0.04 + 0.08 + \ldots + t_{30}$$

The series is geometric, and $a = 0.01$, $r = 2$, and $n = 30$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$ Substitute the values of a, r, and n.

$$S_{30} = \frac{0.01(2^{30} - 1)}{2 - 1}$$ Simplify.

$$= 0.01(1\ 073\ 741\ 823)$$

$$= 10\ 737\ 418.23$$

At the end of 30 days, the grand prize winner would have \$10 737 418.23. Option B offers the greatest grand prize.

Example 3

Find the sum of the geometric series $\frac{1}{16} + \frac{1}{4} + 1 + 4 + \ldots + 65\,536$.

Solution

First find the number of terms in the geometric series, where $t_n = ar^{n-1}$.

Let $t_n = 65\,536$, $a = \frac{1}{16}$, and $r = 4$. Solve for n.

$$65\,536 = \frac{1}{16}(4)^{n-1}$$ Multiply both sides by 16.

$$1\,048\,576 = 4^{n-1}$$ Use trial and error to rewrite 1 048 576 as a power of 4.

$$4^{10} = 4^{n-1}$$ Since the bases are the same, the exponents must be equal.

$$\therefore \quad 10 = n - 1$$

$$11 = n$$

Determine the sum.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$ Substitute $a = \frac{1}{16}$, $r = 4$, and $n = 11$.

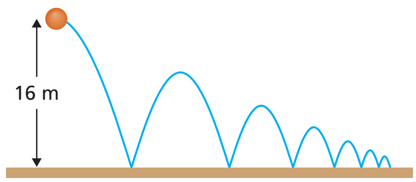
$$S_{11} = \frac{\frac{1}{16}(4^{11} - 1)}{4 - 1}$$ Simplify.

$$= \frac{\frac{1}{16}(4\,194\,303)}{3}$$

$$= \frac{4\,194\,303}{16 \times 3}$$

$$= 87\,381.3125$$

Example 4



Amy drops a ball from a height of 16 m. Each time the ball touches the ground, it bounces up to $\frac{5}{8}$ of the maximum height of the previous bounce. Determine the total vertical distance the ball has travelled when it touches the ground on the seventh bounce. Express your answer to two decimal places.

Solution

Calculate the total vertical distance the ball has travelled by finding the sum of the downward distances and the sum of the upward distances. The upward vertical distance is the same as the downward vertical distance for each bounce. Therefore, the total vertical distance travelled is twice the sum of the downward distances, less 16 m, which is the height from which the ball is dropped. The sum of the downward distances is S_7, the sum of the geometric sequence, with $a = 16$, $r = \frac{5}{8}$, and $n = 7$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$ Substitute the values for a, r, and n.

$$S_7 = \frac{16\left[\left(\frac{5}{8}\right)^7 - 1\right]}{\frac{5}{8} - 1}$$ Simplify.

$$\doteq \frac{-15.403\,953\,55}{-0.375}$$

$$= 41.08$$

The total downward distance is about 41.08 m and the total upward distance is 41.08 − 16 = 25.08 m. The total vertical distance travelled by the ball is 66.16 m.

Practise, Apply, Solve 2.3

A

1. **i.** Determine whether each of the following series is arithmetic, geometric, or neither.

ii. For each series that is geometric, determine the common ratio.

(a) $400 + 200 + 100 + 50 + \ldots$

(b) $15\,000 + 12\,000 + 9600 + 7680 + \ldots$

(c) $24 - 28 + 32 - 36 + 40 - \ldots$

(d) $-3000 + 4500 - 6750 + 10\,125 - \ldots$

(e) $\frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \ldots$

(f) $12 + 16 + 21 + 27 + 34 + \ldots$

2. For each of the following geometric series, determine

i. the general term, t_n

ii. the general sum, S_n

iii. S_8 to two decimal places, where appropriate

(a) $2 + 6 + 18 + 54 + \ldots$

(b) $1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \ldots$

(c) $6 - 12 + 24 - 48 + \ldots$

(d) $81 + 27 + 9 + 3 + \ldots$

(e) $0.4 + 0.04 + 0.004 + 0.0004 + \ldots$

(f) $8 - 8 + 8 - 8 + \ldots$

3. For each of the given geometric series, find the indicated sum. Give your answers to two decimal places, where appropriate.

(a) S_7; $5 - 10 + 20 - \ldots$

(b) S_{10}; $54 + 18 + 6 + \ldots$

(c) S_8; $-300 + 4500 - 67\,500 + \ldots$

(d) S_9; $2 + 2\sqrt{2} + 4 + 4\sqrt{2} + \ldots$

(e) S_n; $1 + x + x^2 + \ldots$

(f) S_n; $5w + 10w^2 + 20w^3 + \ldots$

4. **Knowledge and Understanding:** For the geometric series $6 - 18 + 54 - \ldots$, find

(a) the eighth term

(b) the sum of the first eight terms

B

5. Evaluate each geometric series.

(a) $7 + 14 + 28 + \ldots + 3584$

(b) $-3 - 6 - 12 - 24 - \ldots - 768$

(c) $1 + \frac{5}{2} + \frac{25}{4} + \ldots + \frac{15\,625}{64}$

(d) $96\,000 - 48\,000 + 24\,000 - \ldots + 375$

(e) $1000 + 1000(1.06) + 1000(1.06)^2 + \ldots + 1000(1.06)^{12}$

6. The fifth term of a geometric series is 405 and the sixth term is 1215. Find the sum of the first nine terms.

7. A large school board established a phone tree to contact all of its employees in case of emergencies or inclement weather. Each of the three superintendents calls three employees who each in turn calls three other employees, and so on. How many rounds of phone calls are needed to notify all 9840 employees?

8. **Communication:** Ed begins working as a reporter for a local newspaper. He earns \$1200 for the first month. Each subsequent month, his pay increases by 10%. Describe two different methods for calculating Ed's total pay for the last six months of his first year.

9. Moira wants to share a joke with her friends by e-mail. She sends an e-mail to five friends and asks them to forward her e-mail to five other people, and so on.

(a) Draw a tree diagram to represent the first three rounds of e-mails.

(b) No one receives two copies of the joke. How many people will receive an e-mail of the joke

i. for the first round of e-mails?

ii. for the second round of e-mails?

iii. for the third round of e-mails?

(c) Write an equation to represent the total number of people who receive the e-mail after n rounds of e-mails.

(d) Determine the total number of people who receive the e-mail after eight rounds. What is the likelihood that this event would occur? Justify your answer.

10. When you shut off a circular saw, it continues to turn for a while. Each second, the speed or revolutions per second, r/s, is $\frac{2}{3}$ of the speed of the previous second. At the beginning of the ninth second, the saw has turned a total of 258 times. What was the speed of the saw at the beginning of the first second when it was first shut off? Express your answer to one decimal place.

11. Roger just received his first annual pension cheque of \$19 500. Each subsequent year, the value of the cheque will be 1.02 times the previous year's cheque, to account for 2% inflation.

(a) How much can Roger expect his seventh cheque to be worth?

(b) Determine the total amount he will have received after his tenth cheque.

12. **Application:** A new computer software company earns a profit of \$245 000 in its first year. The company expects the profit to increase by 15% each year for each subsequent year.

(a) What profit can the company expect to earn in its seventh year?

(b) Find the total profit the company will earn in its first ten years.

13. A super ball is dropped from a height of 15 m. After each bounce, the maximum height of the ball is 70% of the ball's maximum height of the previous bounce. What is the total vertical distance that the ball has travelled when it touches the ground after the fifth bounce? Express your answer to two decimal places.

14. A community group has a telethon each year, which is aired on the community cable channel. This year, \$4500 was raised. The fundraisers wish to increase the money raised by 12% each year.

(a) How much would they need to raise from the telethon five years from now to meet their goal?

(b) How much could the fundraisers expect to raise in total after seven years?

15. **Thinking, Inquiry, Problem Solving:** The sum of the terms of any general geometric series is $S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}$. Multiply S_n by the common ratio, r, to obtain an expression for $rS_n - S_n$. Use this expression to prove that $S_n = \frac{a(r^n - 1)}{r - 1}$.

16. A sweepstakes has \$4 000 000 in prizes. The first ticket drawn wins \$15, the second ticket drawn wins \$45, the third ticket drawn wins \$135, and so on.

(a) How many tickets can be drawn without giving away more than the allotted prize money?

(b) How much money is left after all the prizes are awarded?

17. **Check Your Understanding**

(a) Use the method in the Think, Do, Discuss of this section to prove that the sum of the series 2 + 8 + 32 + 128 + 512 + 2048 + 8192 is 10 922.

(b) Verify your solution using $S_n = \frac{a(r^n - 1)}{r - 1}$.

C

18. In a geometric series, $t_1 = 3$ and $S_3 = 21$.

(a) Write an expression to represent the second and third terms.

(b) Use the expressions that you found in (a) to help you determine the common ratio. Explain how there can be two solutions.

19. Show that the sum of n terms of the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \ldots + t_n$ is always less than 4, where n is any natural number.

20. Neither of these series is arithmetic nor geometric, but, by analyzing their patterns, you can find each sum. Find each sum.

(a) $2 - 4 + 6 - 8 + 10 - \ldots - 100$

(b) $1 + 2 + 4 + 5 + 7 + 8 + \ldots + 95 + 97 + 98$

21. The series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$ is an example of an infinite geometric series.

(a) Determine the sum of this series.

(b) Is it possible to find the sum of *any* infinite geometric sequence? Explain.

(c) Under what conditions is it possible to find the sum of an infinite geometric sequence?

The Chapter Problem—Financial Planning

In this section, you studied geometric series. Apply what you learned to answer these questions about the Chapter Problem on page 106.

CP2. Bart's education fund earns interest at 6%/a, compounded monthly. Find the value of the first \$25 deposit after

(a) 1 month (b) 2 months (c) 3 months

(d) 4 months (e) 36 months

CP3. Show why the sequence of the monthly values of the \$25 deposit is a geometric sequence. Determine an expression for t_n, the value of the first deposit after n months.

CP4. Find the sum of the first eight terms of the sequence in CP3.

Did You Know?

The mighty pyramids of Egypt were built thousands of years ago. But when exactly? In *Nature* magazine, Kate Spence has suggested an answer to this question. Spence begins with the fact that one side of the Great Pyramid of Cheops is off true north by exactly 0.05°. The Egyptians did not have compasses, so they may have used the stars to orient the pyramid. Over many centuries, the positions of the stars change because the Earth wobbles slightly on its axis. Spence has shown that in 2478 BCE a straight line drawn between the stars Kochab and Mizar would have been off true north by exactly 0.05°. Thus, Spence concludes that the Great Pyramid was begun in 2478 BCE.