2.1 Arithmetic Series

Over 200 years ago, a teacher in Germany wanted to keep his students busy for a while, so he asked them to add all the whole numbers from 1 to 100. Within a few moments, ten-year-old Karl Friedrich Gauss had the correct answer, while his classmates struggled with the problem for several hours. (They all got the wrong answer.)

Here is Gauss's solution. Suppose you write out the terms of the sum horizontally. Now write the sum in reverse order, underneath the first line. Add each vertical pair of terms.

$$\begin{array}{rrrrrrrrr}
1\, + & 2\, + & 3\, + & 4\, + & 5\, + & \ldots\, + & 98\, + & 99\, + & 100 \\
100\, + & 99\, + & 98\, + & 97\, + & 96\, + & \ldots\, + & 3\, + & 2\, + & 1 \\
\hline
101\, + & 101\, + & 101\, + & 101\, + & 101\, + & \ldots\, + & 101\, + & 101\, + & 101
\end{array}$$

$101 \times 100 = 10\,100$ The sum has 100 terms, so multiply 101 by 100.

$10\,100 \div 2 = 5050$ Each term of the sum from 1 to 100 appears twice, so divide the result by 2.

Therefore $1 + 2 + 3 + \ldots + 100 = 5050$.

You will use this method in the following problem.

An auditorium is being built in the shape of a semicircle with 12 seats in the first row, 15 seats in the second row, 18 seats in the third row, 21 seats in the fourth row, and so on.

Think, Do, Discuss

1. Recall working with sequences in Chapter 1. What type of sequence describes the seating? Explain. How many seats are in the tenth row? Find the general term, t_n, that represents the number of seats in the nth row.
2. How would you find the total number of seats in the first seven rows? Would this method be efficient for finding the number of seats in the first 30 rows? Explain.
3. The sum of the terms of a sequence is called a **series**. The sum of the first seven rows of seats is S_7, where $S_7 = t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + t_7$. The sum of the first 30 rows of seats is S_{30}. Express S_7 as the sum of seven terms, using the general term, t_n, you found in step 1. Write this sum in your notebook.

4. Use Gauss's method to guide you through the following steps.

(a) Rewrite the sum for S_7 in reverse order so that t_7 is below t_1, t_6 is below t_2, and so on.

(b) Add each pair of terms, for example, $t_7 + t_1$. What is the sum of each pair?

(c) How many pairs of terms are there? Calculate the sum of the pairs of terms and let this sum be equal to $2S_7$.

(d) What must you do to $2S_7$ to determine S_7, which is the number of seats in the first seven rows? Verify your solution by adding the terms in S_7.

5. How many seats are in the 30th row? Use what you learned in the previous steps to find the total number of seats in the first 30 rows.

6. Suppose you know the first term and the last term of an arithmetic sequence. Suggest a formula for finding S_n, the sum of any **arithmetic series**.

Focus 2.1

Key Ideas

- A **series** is the sum of the terms of a sequence. The sum of the first n terms of a sequence is S_n, where

$$S_n = t_1 + t_2 + t_3 + t_4 + \ldots + t_{n-1} + t_n$$

- An arithmetic sequence has the general term $t_n = a + (n - 1)d$, where a is the **first term** and d is the **common difference** between terms. The sum of this sequence is called an **arithmetic series** and is

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + [a + (n - 2)d] + [a + (n - 1)d]$$

- Find the sum of the series, S_n, by adding the first term, t_1, and the last term, t_n, together, multiplying by the number of terms, n, and then dividing the result by 2.

- The sum of the n terms of an arithmetic series is

$$S_n = \frac{n(t_1 + t_n)}{2}$$

- Substitute a for t_1 and substitute $a + (n - 1)d$ for t_n in $S_n = \frac{n(t_1 + t_n)}{2}$. The result produces the sum of the first n terms of an arithmetic series,

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

Use this formula when you know the first term of the series, the common difference, and the number of terms.

Example 1

For the series 2 + 11 + 20 + 29 + … , find

(a) t_{20} (b) S_{20}

Solution

(a) The series is arithmetic, and $a = 2$, $d = 9$, and $n = 20$.

$$t_n = a + (n - 1)d$$

Therefore, $t_{20} = 2 + 19(9) = 173$.

(b) Since the first and the last terms are known, use $S_n = \frac{n(t_1 + t_n)}{2}$ to find S_{20}.

$$\begin{aligned} S_{20} &= \frac{20(2 + 173)}{2} \\ &= 10(175) \\ &= 1750 \end{aligned}$$

Example 2

A military unit buys 10 spare parts during the first month of a contract, 15 spare parts in the second month, 20 spare parts in the third month, 25 spare parts in the fourth month, and so on. The acquisitions officer wants to know the total number of spare parts the unit will have after 24 months.

Solution

The sequence of the number of spare parts bought each month is 10, 15, 20, 25, … . For this arithmetic sequence, $a = 10$ and $d = 5$. Find the sum of the first 24 terms.

$$S_n = \frac{n}{2}[2a + (n - 1)d] \qquad \text{Substitute } a = 10,\ d = 5, \text{ and } n = 24.$$

$$S_{24} = \frac{24}{2}[2(10) + (24 - 1)(5)] \qquad \text{Simplify.}$$

$$= 12(20 + 115)$$

$$= 1620$$

The unit will have a total of 1620 spare parts after 24 months.

Example 3

Samantha deposited \$128 in her account. Each week, she deposits \$7 less than the previous week until she makes her last deposit of \$9. Find the total value of her deposits.

Solution

The arithmetic series 128 + 121 + 114 + … + 16 + 9 represents the sum of her weekly deposits. To calculate the sum, first find the number of terms in the series.

$$
\begin{aligned}
t_n &= a + (n - 1)d && \text{Substitute } a = 128,\ d = -7, \text{ and } t_n = 9.\\
9 &= 128 + (n - 1)(-7) && \text{Expand.}\\
9 &= 128 - 7n + 7 && \text{Solve for } n.\\
-126 &= -7n\\
18 &= n
\end{aligned}
$$

There are 18 terms in the series.

Now find the sum.

$$
\begin{aligned}
S_n &= \frac{n(t_1 + t_n)}{2} && \text{Substitute } n = 18,\ t_1 = 128, \text{ and } t_n = 9.\\
S_{18} &= \frac{18(128 + 9)}{2} && \text{Simplify.}\\
&= 1233
\end{aligned}
$$

Samantha deposited \$1233 in her account.

Example 4

The fifth term of an arithmetic series is 9, and the sum of the first 16 terms is 480. Find the first three terms of the series.

Solution

For an arithmetic sequence, $t_n = a + (n - 1)d$. In this case, $n = 5$ and $t_5 = 9$.

Therefore, $9 = a + 4d$ ①

For an arithmetic series, $S_n = \frac{n}{2}[2a + (n - 1)d]$. In this case, $n = 16$ and $S_{16} = 480$.

$$
\begin{aligned}
480 &= \frac{16}{2}(2a + 15d)\\
480 &= 8(2a + 15d)\\
60 &= 2a + 15d \quad ②
\end{aligned}
$$

To find a and d, solve the linear system of equations ① and ②.

$$
\begin{aligned}
9 &= a + 4d && ①\\
60 &= 2a + 15d && ② && \text{Multiply equation ① by 2 to obtain ③.}\\
18 &= 2a + 8d && ③\\
60 &= 2a + 15d && ② && \text{Subtract ② from ③.}\\
-42 &= -7d\\
6 &= d
\end{aligned}
$$

Substitute $d = 6$ in equation ① to obtain the value of a.

$$
\begin{aligned}
9 &= a + 4(6) \quad ①\\
a &= -15
\end{aligned}
$$

The first three terms of the series are -15, -9, and -3.

Practise, Apply, Solve 2.1

A

1. Determine whether each series is arithmetic.

(a) $5 + 8 + 11 + 14 + 17 + \dots$

(b) $1 + 2 + 3 + 5 + 8 + 13 + \dots$

(c) $-83 - 77 - 71 - 65 - 59 - \dots$

(d) $a + 2a + 4a + 8a + 16a + \dots$

2. For each series, calculate t_{10} and S_{10}.

(a) $2 + 10 + 18 + 26 + \dots$

(b) $100 + 85 + 70 + 55 + \dots$

(c) $-18 - 11 - 4 + 3 + \dots$

(d) $\frac{1}{6} + \frac{1}{2} + \frac{5}{6} + \frac{7}{6} + \dots$

3. **Communication:** You could add whole numbers from 1 to 10 by evaluating $\frac{10 \times 11}{2}$, and you could add the whole numbers from 1 to 50 by evaluating $\frac{50 \times 51}{2}$. How would you add the whole numbers from 1 to 200? Explain how the terms in these expressions are obtained.

4. Find the sum of the whole numbers from

(a) 1 to 30 (b) 1 to 60

(c) 1 to 1000 (d) 1 to 24

5. Find the sum of each arithmetic series, given the first and the last terms.

(a) $t_1 = 7$ and $t_{12} = 51$ (b) $t_1 = 88$ and $t_{15} = 4$

(c) $t_1 = -784$ and $t_{20} = 869$ (d) $t_1 = -2$ and $t_{30} = 85$

6. Find the sum of each series.

(a) $3 + 7 + 11 + \dots + t_{15}$ (b) $11 + 22 + 33 + \dots + t_{20}$

(c) $4 - 1 - 6 - \dots - t_{27}$ (d) $\frac{1}{2} + \frac{5}{8} + \frac{3}{4} + \dots + t_{14}$

B

7. **Knowledge and Understanding:** Find S_{21} for the series $2.8 + 3.2 + 3.6 + 4.0 + \dots$.

8. Find the sum of each arithmetic series.

(a) $13 + 24 + 35 + \dots + 156$

(b) $15 + 11 + 7 + \dots - 37$

(c) $-77 - 70 - 63 - \dots + 252$

(d) $\frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \dots + \frac{5}{3}$

9. In an arithmetic series of 50 terms, the 17th term is 53 and the 28th term is 86. Find the sum of the series.

10. In an arithmetic series of 20 terms, the seventh term is 34 and the ninth term is 48. Find the sum of the series.

11. In an arithmetic series, the 15th term is 43 and the sum of the first 15 terms is 120. Add the first 20 terms of the series.

12. In an arithmetic series, the 12th term is 15 and the sum of the first 15 terms is 105. Add the first three terms of the series.

13. **Application:** An usher was asked to count the number of seats in an auditorium. The auditorium had 25 rows of seats. He counted the 59 seats in the last row at the back and noticed that each subsequent row toward the front row had two fewer seats than the row before it. How many seats are in the auditorium?

14. The sum of the first n odd numbers is 400. What is the value of n?

15. The arithmetic series $5 + 9 + 13 + \ldots + t_n$ has a sum of 945. How many terms does the series have?

16. In a lecture hall, there are 13 seats in the first row. The next six rows increase by two seats each. Each of the remaining rows increases by three seats each. There are 12 rows in all. How many seats are in the lecture hall?

17. A skydiver jumped out of a plane. She fell 4.9 m in the first second, 14.7 m in the second second, 24.5 m in the third second, and so on, in the same pattern, until she opened her parachute. How far did she fall between the eighth second and the thirteenth second?

18. **Check Your Understanding:** Explain, in your own words, when it is better to use $S_n = \frac{n(t_1 + t_n)}{2}$ instead of $S_n = \frac{n}{2}[2a + (n - 1)d]$. What values must you know to find each sum?

C

19. Michelle purchased a used car for $17 000. Once Michelle drove away from the lot, the car's value depreciated by 20%. For each subsequent year, the car's value depreciated by $450. What was the total depreciation after six years?

20. A hockey arena has 10 920 seats. The first row of seats around the rink has 220 seats. The number of seats in each subsequent row increases by 16.

(a) How many rows of seats does the arena have?

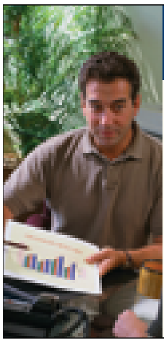
(b) The arena's owners would like to expand the arena by adding four more rows of seats. What will be the new capacity of the arena?

21. Three squares are added to a 1 by 1 square to form a 2 by 2 square. How many squares must be added to the 2 by 2 square to produce a 3 by 3 square? Use an arithmetic series to prove that an n by n square has a total of n^2 squares.

22. **Thinking, Inquiry, Problem Solving:** The sum of the terms of the general arithmetic series is $S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + [a + (n - 2)d] + [a + (n - 1)d]$. Use the method in the Think, Do, Discuss, which begins with rewriting the terms of the series in reverse order, to show that $S_n = \frac{n}{2}[2a + (n - 1)d]$.

23. Evaluate $(1 + 5 + 9 + 13 + \ldots + 201) - (3 + 7 + 11 + 15 + \ldots + 203)$.

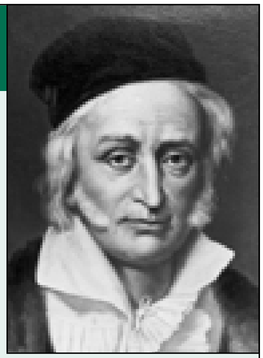
24. Evaluate $300 - 299 + 298 - 297 + 296 - \ldots + 100 - 99$.



The Chapter Problem—Financial Planning

CP1. Visit a local bank or credit union, or search the Internet to obtain information about registered retirement savings plans (RRSPs), guaranteed investment certificates (GICs), registered education savings plans (RESPs), and registered income funds (RIFs). Briefly describe each type of savings plan and explain its purpose.

Karl Friedrich Gauss (1777–1855)



Karl Friedrich Gauss is one of the three greatest mathematicians who ever lived. The other two are Archimides and Isaac Newton. By age 24, Gauss had made essential advances in many areas of math and science, including geometry, number theory, probability theory, astronomy, and electricity. Read about this remarkable man in *Men of Mathematics,* by E.T. Bell, or use the Internet to do research on him. Why is there a statue of Gauss in his birthplace standing on a star with 17 points?