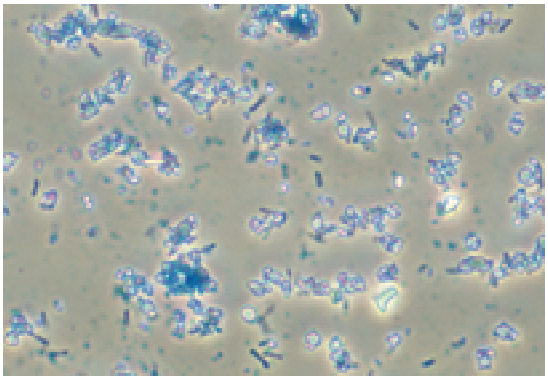
Geometric Sequences 1.7

Part 1: Doubling

Many microorganisms are introduced into our bodies through everyday activities, breathing, eating, and drinking water. Some microorganisms are beneficial and some are not.

One such bacterium is *Escherichia Coli* or *E. coli*. *E. coli* bacteria live in the intestine and help your body to absorb certain vitamins. However, one strain of *E. coli* bacteria is particularly harmful. The rate at which bacteria cells divide and reproduce determines the time it takes for the bacteria to affect your health.



***Escherichia coli* bacteria**

E. coli bacteria are very small. About 50 cells lined up end to end would be roughly as thick as a strand of your hair. As few as 100 cells would cause infection. Each bacterium divides in two about every one-half hour to one hour.

Think, Do, Discuss

1. Assume that 100 *E. coli* bacteria double in number every hour. Copy and complete the table to show how many bacteria are present after each hour for 8 h.

Time (h)	0	1	2	3	4	5	6	7	8
Number of Bacteria	100	200							

2. Write the number of bacteria after each hour for 8 h as a sequence. The first term is 100.
3. Determine the first differences between pairs of consecutive terms of this sequence. Interpret the first differences.
4. Graph this sequence. Describe the rate of change between pairs of consecutive terms.
5. Divide each term, except the first, by the previous term. What do you notice? What is the significance of this value?
6. Rewrite each term of the sequence that you wrote in step 2 as an expression of the original 100 bacteria and the value that you calculated in step 5.
7. Write the general term for this sequence.
8. Determine the number of bacteria after 24 h.

Part 2: Half-life

Half-life is the time required for a radioactive material to decay to one-half of its original mass. Uranium-238, used in nuclear reactors, has a very long half-life, 4.5×10^9 years. Only a small fraction of uranium-238 has decayed since the Earth was formed. However, carbon-11, used in medical applications, has a half-life of only 20 min.

Pickering Nuclear Reactor in Ontario

Think, Do, Discuss

1. After a medical procedure, Harry's body has absorbed 100 mg of carbon-11. Copy and complete the table to show the mass of carbon-11 after each 20-min interval for 3 h.

20-Min Interval	0	1	2	3	4	5	6	7	8	9
Mass of Carbon-11 (mg)	100	50								

2. Write the mass of carbon-11 after each 20-min interval for 3 h as a sequence. The first term is 100.

3. Determine the first differences between pairs of consecutive terms of this sequence. Interpret the first differences.

4. Graph this sequence. Describe the rate of change between pairs of consecutive terms.

5. Divide each term, except the first, by the previous term. What do you notice? What is the significance of this value?

6. Rewrite each term of the sequence that you wrote in step 2 as an expression of the original 100 mg and the value that you calculated in step 5.

7. Write the general term for this sequence.

8. Determine the mass of carbon-11 remaining after 24 h.

Focus 1.7

Key Ideas

- In a **geometric sequence**, the ratio of any term, except the first one, to the previous term, is constant for all pairs of consecutive terms. This constant or **common ratio** is represented by r.

  In the general geometric sequence $t_1, t_2, t_3, t_4, \ldots, t_n, \ldots, r = \frac{t_n}{t_{n-1}}$.

  Here are some examples of geometric sequences.

  $4, 8, 16, 32, 64, \ldots$, and $r = 2$

  $-3, 6, -12, 24, -48, \ldots$, and $r = -2$

  $100, 10, 1, 0.1, 0.01, \ldots$, and $r = 0.1$

  $\frac{1}{2}, \frac{1}{8}, \frac{1}{32}, 1, 128, \ldots$, and $r = \frac{1}{4}$

- The general term of a geometric sequence is

  $t_n = ar^{n-1}$

  where t_n is the nth term,
  a is the first term, or t_1, and
  r is the common ratio.

- If $r > 1$, then the terms increase. If $0 < r < 1$, then the terms decrease.
- The relationship between n and t_n of any geometric sequence is nonlinear.

Example 1

For each sequence,

i. determine if it is arithmetic, geometric, or neither
ii. determine the next three terms
iii. find the general term and use it to determine t_{10}
iv. graph the sequence

(a) 4, 12, 36, 108, … **(b)** 100, 98, 96, 94, …

Solution

(a) **i.** Starting with the second term, divide each term by the term before it.

$$\frac{12}{4} = 3, \frac{36}{12} = 3, \frac{108}{36} = 3$$

This sequence is geometric and has a common ratio of 3.

ii. To find the next three terms, multiply each new term by the common ratio.

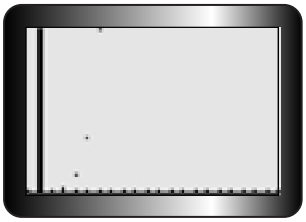
$$108 \times 3 = 324, 324 \times 3 = 972, \text{and } 972 \times 3 = 2916$$

The next three terms are 324, 972, and 2916.

iii. The general term is $t_n = ar^{n-1}$, with $a = 4$ and $r = 3$.

$$\begin{aligned} t_n &= 4(3)^{n-1} \\ t_{10} &= 4(3)^{10-1} \\ &= 4(3)^9 \\ &= 78\ 732 \end{aligned}$$

iv.



(b) **i.** Starting with the second term, subtract each term from the previous term.

$$98 - 100 = -2,\ 96 - 98 = -2,\ 94 - 96 = -2$$

This sequence is arithmetic and has a common difference of -2.

ii. To find the next three terms, add the common difference to each new term.

$$94 + (-2) = 92,\ 92 + (-2) = 90,\ 90 + (-2) = 88$$

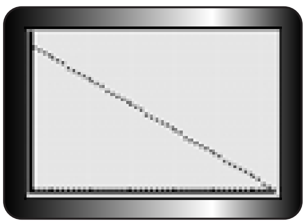
The next three terms are 92, 90, and 88.

iii. The general term is $t_n = a + (n - 1)d$, with $a = 100$ and $d = -2$.

$$\begin{aligned} t_n &= 100 + (n - 1)(-2) \\ &= 100 - 2n + 2 \\ &= 102 - 2n \end{aligned}$$

$$\begin{aligned} t_{10} &= 102 - 2(10) \\ &= 82 \end{aligned}$$

iv.



Example 2

A new provincial lottery offers two choices to the grand prize winner. Option A is a lump-sum payment of \$25 000 000. Option B is a yearly payment on the winner's birthday that starts with \$1 and doubles each year thereafter for the rest of the winner's life. Suppose you win this lottery on your 20th birthday. Which option would you choose?

Solution

The payments under option B are \$1, \$2, \$4, \$8, \$16, … .
This sequence is geometric, with $a = 1$ and $r = 2$.

$$t_n = ar^{n-1}$$
$$t_n = 1(2)^{n-1}$$

Use a table to examine the pattern in more detail.

r (year)	t_n (payment, \$)	r (year)	t_n (payment, \$)	r (year)	t_n (payment, \$)
1	1	10	512	19	262 144
2	2	11	1 024	20	524 288
3	4	12	2 048	21	1 048 576
4	8	13	4 096	22	2 097 152
5	16	14	8 192	23	4 194 304
6	32	15	16 384	24	8 388 608
7	64	16	32 768	25	16 777 216
8	128	17	65 536	26	33 554 432
9	256	18	131 072	27	67 108 864

Clearly, at the end of 27 years, option B surpasses option A in winnings. Keep in mind that the total earnings under option B is the sum of all the terms in the sequence. Consider the payment on the 60th birthday by itself.

$$\begin{aligned} t_{60} &= 1(2)^{60-20-1} \\ &= 1(2)^{39} \\ &= 5.497\ 558\ 139 \times 10^{11} \end{aligned}$$

The payment on the 60th birthday would be \$549 755 813 900.

Example 3

Since 1967, the average annual salary of major league baseball players has risen by about 17% each year. In 1967, the average annual salary was \$19 000.

(a) Predict the average annual salary of a baseball player in 2007.

(b) Verify your answer using graphing technology.

Solution

(a) Each year, the new average annual salary is the salary of the previous year, plus an increase of 17% of that amount, or 117% of the previous salary.

For 1968, $19\ 000 \times 1.17 = 22\ 230$
For 1969, $22\ 230 \times 1.17 = 26\ 009.10$

Year	1967	1968	1969	1970	1971	1972
Average Salary ($)	19 000	22 230	26 009.10	30 430.65	35 603.86	41 656.51

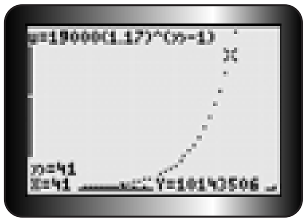
The average salary each year is a geometric sequence, where $a = 19\ 000$ and $r = 1.17$. For a geometric sequence, $t_n = ar^{n-1}$. In this case, $t_n = 19\ 000(1.17)^{n-1}$ and $n = 1$ corresponds to 1967. Therefore, in 2007,

$$\begin{aligned} n &= 2007 - (1967 - 1) \\ &= 41 \end{aligned}$$

$$\begin{aligned} t_{41} &= 19\ 000(1.17)^{41-1} \\ &= 19\ 000(533.868\ 712\ 7) \\ &= 10\ 143\ 505.54 \end{aligned}$$

The model predicts an average annual salary of $10 143 505.54 in 2007.

(b) Using the TI-83 Plus calculator, select the sequence graphing mode ([MODE]) and enter the general term in the sequence editor ([Y=]). Adjust the window ([WINDOW]) and [GRAPH]. [TRACE] to find the value of the 41st term.



Example 4

The half-life of iodine-131 is eight days.

(a) What will remain of 12 mg of iodine-131 after 112 days?

(b) Verify your answer using graphing technology.

Solution

(a)

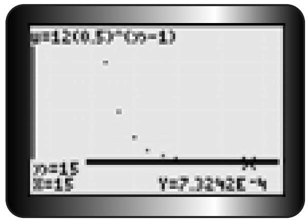
Eight-Day Interval	0	1	2	3
Mass Remaining (mg)	12	$12 \times 0.5 = 6$	$6 \times 0.5 = 3$	$3 \times 0.5 = 1.5$

The mass remaining after each interval forms the sequence 12, 6, 3, 1.5, … . This sequence is geometric, with $a = 12$ and $r = 0.5$. For a geometric sequence, $t_n = ar^{n-1}$. In this case, $t_n = 12(0.5)^{n-1}$ and $n = 1$ corresponds to the initial mass of 12 mg. In 112 days, the number of eight-day intervals is $\frac{112}{8}$ or 14. Since $t_1 = 12$, then t_{15} is the mass remaining after the 14th interval.

$$\begin{aligned} t_{15} &= 12(0.5)^{15-1} \\ &= 12(0.000\ 061\ 035) \\ &= 0.000\ 732\ 42 \end{aligned}$$

After 112 days, 0.000 732 42 mg of iodine-131 will remain.

(b) Using the TI-83 Plus calculator, select the sequence graphing mode ([MODE]) and enter the general term in the sequence editor ([Y=]). Adjust the window ([WINDOW]) and [GRAPH]. [TRACE] to find the value of the 15th term.



Practise, Apply, Solve 1.7

A

1. **i.** Identify which sequences are geometric.
ii. For each geometric sequence, determine the common ratio.

(a) 4, 12, 36, 108, …
(b) −5, 15, −45, 135, …
(c) 17, 24, 31, 38, …
(d) 2, 8, 32, 128, …
(e) 56, 45, 34, 23, …
(f) $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

2. **i.** Determine which general terms represent geometric sequences.
ii. Determine the common ratio of each geometric sequence.

(a) $t_n = 3^{n-1}$
(b) $t_n = (n + 3)(n - 5)$
(c) $t_n = -2n^2 - 5n + 1$
(d) $t_n = 2(6)^{n+1}$
(e) $t_n = \frac{1}{3n - 2}$
(f) $t_n = 5(-1)^n$

3. For each of the following geometric sequences, determine

i. the common ratio **ii.** the general term **iii.** t_8

(a) 3, 15, 75, 375, …
(b) −12, −144, −1728, −20 736, …
(c) 4, 2, 1, $\frac{1}{2}$, …
(d) 6, −12, 24, −48, …
(e) 0.2, 0.02, 0.002, 0.0002, …
(f) 5, 5, 5, 5, …

4. Determine the recursive formula for each sequence in question 3.

5. Find the general term of the geometric sequence in which

(a) the first term is 3 and the common ratio is 7
(b) the first term is −4 and the common ratio is $-\frac{1}{4}$
(c) the first term is 125 and the common ratio is −0.2
(d) the first term is −6 and the common ratio is 3

6. **i.** Determine whether each sequence is arithmetic, geometric, or neither.
ii. If a sequence is arithmetic or geometric, then find t_n.

(a) 30, 40, 50, 60, …

(b) 4, 16, 64, 256, …

(c) 2, 6, 7, 21, 22, …

(d) 30, 6, 1.2, 0.24, …

(e) 145, 130, 115, 100, …

(f) 1.21, 1.44, 1.69, 1.96, …

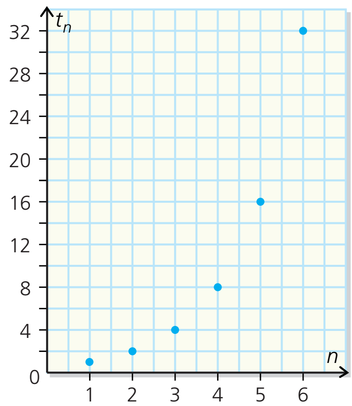
B

7. For each geometric sequence, find

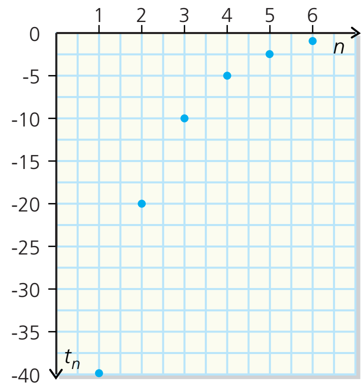
(a) t_n, the general term

(b) t_{10}

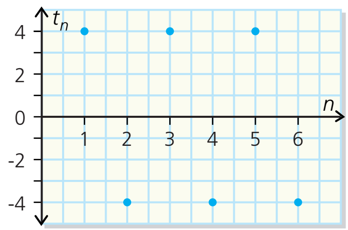
i.



ii.



iii.



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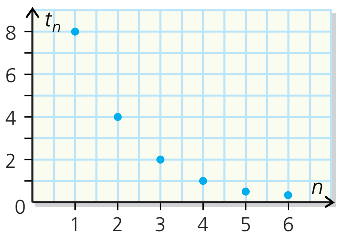


8. Graph each sequence.

(a) $t_n = 5^{n-1}$

(b) $t_n = -2(3)^{n-1}$

(c) $t_n = 10\left(\frac{1}{2}\right)^n$

(d) $t_n = 3(-2)^{n+1}$

(e) $t_n = 4(5)^{n+2}$

(f) $t_n = (2)^{2n}$

(g) $t_n = 0.5(4)^{n-2}$

(h) $t_n = -(10)^{n+1}$

9. **Knowledge and Understanding:** Consider the sequence $1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots$.

(a) Determine the general term of this sequence.

(b) Find t_8.

10. Suki bought a car for $28 500 and its value depreciates by 12% each year.

(a) Write the sequence of the car's value at the end of each year for the next five years. Start with the purchase price.

(b) Determine the general term of this sequence.

(c) Determine the car's value at the end of the eighth year.

11. Todd accepts a job with a graphic design firm. His starting salary is \$34 000, and each year he will receive an annual increase of 2.5%.

(a) Write the sequence of his annual salary for the next five years. Start with Todd's initial salary.

(b) Determine the general term of this sequence.

(c) Determine Todd's annual salary at the end of his tenth year.

12. A rare coin is bought at an antique auction in 1998 for \$500. Each year, its value appreciates by 6% of its purchase price. Determine the coin's value in 2010.

13. The half-life of a radioactive substance is 10 min. How much of 500 g of this substance will remain after 8 h?

14. A colony of bacteria doubles in number every 20 min. The initial population of the colony is 20 bacteria. How many bacteria are there after 4 h?

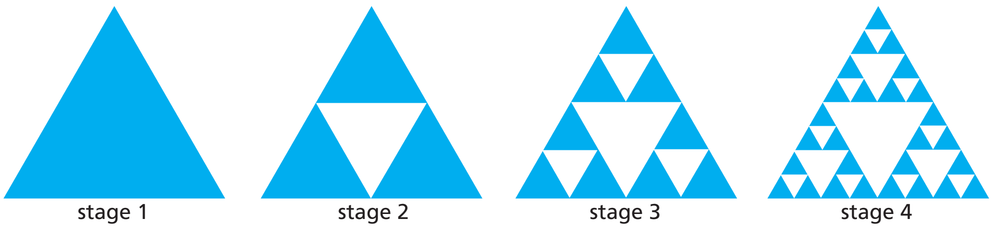
15. Since 1900, the world's population has grown at an average rate of 1.35% per year. In 1900, the world's population was 1.65 billion.

(a) Determine the general term of the sequence that models this relationship.

(b) Use the model to predict the world's population in 2010.

16. **Communication:** A printing company buys a photocopying machine for \$250 000. The machine's value depreciates at a rate of 25% per year. Explain how this situation involves a geometric sequence.

17. Examine each stage.



(a) Find the first four terms of the sequence that represents the number of shaded triangles at each stage.

(b) How many shaded triangles does stage 8 have?

(c) Find the first four terms of the sequence that represents the perimeter of the shaded figure at each stage. Use the fact that the largest triangle is equilateral, and each side is one unit long.

(d) Determine the perimeter of the shaded figure in stage 7.

18. Write the recursive formula for any geometric sequence. Let a be the first term and r be the common ratio.

19. **Application:** The value of a new sports utility vehicle depreciates at a rate of 8% per year. If the vehicle was bought for \$45 000, when is it worth less than 40% of its original value?

20. For each geometric sequence, find a, r, and t_n.

(a) $t_6 = -4$, $t_7 = -20$ (b) $t_2 = 8$, $t_4 = 32$ (c) $t_4 = 54$, $t_7 = 1458$

21. **Check Your Understanding**

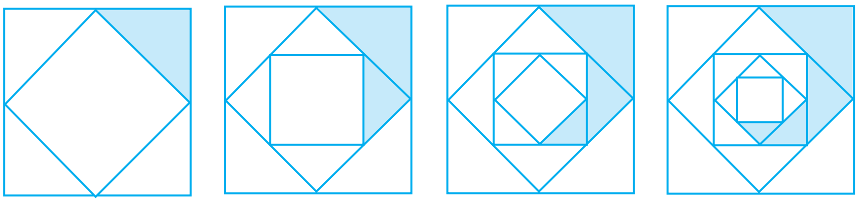
(a) Determine t_n and t_{10} for the sequence 4, 28, 196, 1372, … .

(b) The third and fourth terms of a geometric sequence are 12 and 4, respectively. Determine the first and second terms. Explain your answer.

C

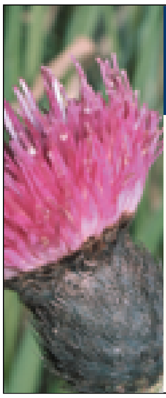
22. Write the first five terms of a sequence that is both arithmetic and geometric.

23. **Thinking, Inquiry, Problem Solving:** A square measures 12 cm by 12 cm. Connect the midpoints of the square to make a smaller square and four triangles. Shade one triangle. Repeat these steps five more times. At this stage, determine the total area of the shaded region.



24. The first three terms of the sequence 8, a, b, 36 form an arithmetic sequence, but the last three terms form a geometric sequence. Find all the possible values of a and b.

25. Find the tenth term of the sequence $\frac{a^2}{b}$, $-a$, b, $\frac{-b^2}{a}$, … .



The Chapter Problem—Controlling Non-Native Plant Populations

In this section, you have studied geometric sequences. Apply what you have learned to answer these questions about the Chapter Problem on page 12.

CP13. Are either of the plant or seed sequences you found in CP9 and CP10 geometric? Explain.

CP14. Find the general term for the plant sequence.

CP15. Find the general term for the seed sequence.