Arithmetic Sequences 1.6

Part 1: The Days of a Month

Here is a typical calendar for October. Do the dates have any predictable patterns?

October						
Sunday	*Monday*	*Tuesday*	*Wednesday*	*Thursday*	*Friday*	*Saturday*
		1	2	3	4	5
6	7	8	9	10	11	12
13	14	15	16	17	18	19
20	21	22	23	24	25	26
27	28	29	30	31		

Think, Do, Discuss

1. Write the sequence of dates for the last week of October. Start with Sunday's date.
2. How many terms are in this sequence? Describe the pattern in this sequence.
3. Extend the pattern and write the next five terms of this sequence. (Assume that the dates may be more than 31.)
4. Determine the **first differences** between pairs of consecutive terms of this sequence. What do you notice? What type of relationship exists between n and the value of the term?
5. Graph this sequence. What is the rate of change between each pair of consecutive terms in this sequence?
6. Determine the general term for this sequence.
7. Write the sequence of dates for all the Tuesdays of the month.

8. Repeat steps 2 to 6 for the sequence in step 7.
9. Describe how to locate, in the calendar, a sequence where the rate of change between each pair of consecutive terms is 8.
10. Describe how to locate, in the calendar, a sequence where the rate of change between each pair of consecutive terms is 6.

Part 2: Simple Interest

Ed is planning to buy a sofa, not with cash, but through a financing plan. After Ed makes a down payment, he must pay a finance charge, which is a percent of the balance owing and depends on the length of time that Ed will take to pay the balance owing. The finance charge is an example of **simple interest**.

Interest is the cost of using money. A bank pays interest for using money in, for example, savings accounts. You pay interest when you use or borrow money.

Interest is calculated as a percent of an amount of money, called the **principal**, which is used for a specific period. The interest is calculated at the end of a period and is added to the principal. However, at the end of the next period, simple interest is again calculated, but **only** on the principal. As a result, the interest earned each period is constant, and the balance grows by the same amount each year.

The formula for **simple interest** is

$$I = Prt$$

where I is the simple interest,
P is the principal (the amount of money borrowed or invested),
r is the annual interest rate, and
t is the time in years.

The balance in an account or the balance owing on a loan at the end of t years, T, is the sum of the original principal, P, and the interest earned or charged, I.

$$\begin{aligned} T &= P + I \\ &= P + Prt \\ &= P(1 + rt) \end{aligned}$$

Examining Simple Interest

On the day Juan is born, his grandparents deposit \$10 000 in a savings account that pays simple interest. The annual interest rate is 5%. If they do not make any other deposits, what is the balance in the account on the day Juan turns 18?

Think, Do, Discuss

1. Calculate the interest earned at the end of the first year.
2. Determine the balance in the account on Juan's first birthday.
3. Calculate the interest earned at the end of the second year.
4. Determine the balance in the account on Juan's second birthday.
5. Write the sequence that represents the balance in the account on each birthday from the 1st to the 18th.
6. What is the balance in the account on Juan's 18th birthday? How much of the balance is interest?
7. Explain the pattern in this sequence. What is the rate of change between each pair of consecutive year-end balances?
8. Graph this sequence.
9. Is the growth of the balance in the account linear or nonlinear? Explain.
10. Let n represent the number of years. Determine
    (a) the general term (b) the recursive formula for this sequence
11. Suppose Juan leaves the money in the account until he retires at 65. What would be the balance in the account on his 65th birthday?
12. Mary deposits \$5000 in an account that earns simple interest at a rate of 8%/a. If she makes no other deposits, determine the sequence that represents the balance at the end of each year for ten years. Determine the general term of this sequence.

Focus 1.6

Key Ideas

- In an **arithmetic sequence**, the **common difference**, d, is a constant value. For the general arithmetic sequence $t_1, t_2, t_3, t_4, \ldots, t_n, \ldots, d = t_n - t_{n-1}$. Here are some examples of arithmetic sequences:

  2, 6, 10, 14, 18, … , and $d = 4$

  5, 2, −1, −4, −7, … , and $d = -3$

  0, 0.25, 0.5, 0.75, 1.0, 1.25, … , and $d = 0.25$

  $\frac{1}{8}, \frac{1}{2}, \frac{7}{8}, 1\frac{1}{4}, 1\frac{5}{8}, \ldots$, and $d = \frac{3}{8}$

- The common difference in an arithmetic sequence corresponds to the rate of change between any two consecutive terms of the sequence. The first differences are constant. Therefore, the terms of an arithmetic sequence change at a constant rate. So, in any arithmetic sequence, the relationship between n and t_n is linear.

- The **general term** of an arithmetic sequence is

$$t_n = a + (n - 1)d$$

where t_n is the nth term,
a is the first term, or t_1, and
d is the common difference.

- An example of an arithmetic sequence is the annual balance in an account that pays simple interest. This investment grows at a constant or linear rate.

Example 1

Identify which sequences are arithmetic. If a sequence is arithmetic, determine the common difference and the general term.

(a) $-9, -6, -3, 0, \ldots$ **(b)** $11, 22, 44, 88, \ldots$

Solution

For each sequence, find the differences between each pair of consecutive terms. If the differences are the same, the sequence is arithmetic.

(a) For this sequence,

$$-6 - (-9) = 3,\ -3 - (-6) = 3, \text{ and } 0 - (-3) = 3$$

Because the common difference is 3, the sequence is arithmetic. Now find the general term.

$$\begin{aligned} t_n &= a + (n - 1)d && \text{Substitute known values.} \\ &= -9 + (n - 1)3 \\ &= -9 + 3n - 3 \\ &= 3n - 12 \end{aligned}$$

The general term is $t_n = 3n - 12$.

(b) For this sequence,

$$22 - 11 = 11,\ 44 - 22 = 22, \text{ and } 88 - 44 = 44$$

Because the differences are not common, this sequence is not arithmetic, and the general term cannot be found.

Example 2

Determine the 25th term of the sequence $-17, -10, -3, 4, \ldots$.

Solution

For this sequence,

$$-10 - (-17) = 7,\ -3 - (-10) = 7, \text{ and } 4 - (-3) = 7$$

This sequence is arithmetic, with $d = 7$ and $a = -17$. Find the general term.

$$\begin{aligned} t_n &= a + (n - 1)d && \text{Substitute.} \\ &= -17 + (n - 1)7 && \text{Expand.} \\ &= -17 + 7n - 7 && \text{Simplify.} \\ &= -24 + 7n \end{aligned}$$

Substitute $n = 25$ in the general term to find the 25th term.

$$\begin{aligned} t_n &= -24 + 7n \\ t_{25} &= -24 + 7(25) \\ &= 151 \end{aligned}$$

Example 3

Determine the number of terms in the finite arithmetic sequence 3, 15, 27, … , 495.

Solution

In this arithmetic sequence, $a = 3$ and $d = 12$. Substitute to find the general term.

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= 3 + (n - 1)12 \\ &= 3 + 12n - 12 \\ &= -9 + 12n \end{aligned}$$

To find the number of terms, substitute the value of the last term for t_n and solve for n.

$$\begin{aligned} t_n &= -9 + 12n \\ 495 &= -9 + 12n \\ 495 + 9 &= 12n \\ 504 &= 12n \\ 42 &= n \end{aligned}$$

This sequence has 42 terms.

Example 4

Irma deposits \$750.00 in a savings account that earns simple interest. This table shows the balance at the end of each year for four years.

Year	1	2	3	4
Year-End Balance ($)	783.75	817.50	851.25	885.00

Determine the following.

(a) the annual interest rate

(b) the general term for the balance at the end of the nth year

(c) the balance at the end of the 15th year

Solution

(a) Each year, the account earns the same amount of interest. This amount is

$$\$783.75 - \$750.00 = \$33.75$$

In this case, $I = \$33.75$, $P = \$750$, and $t = 1$. Use the formula $I = Prt$ to find r.

$$\begin{aligned} I &= Prt && \text{Substitute.} \\ 33.75 &= 750r(1) \\ 33.75 &= 750r && \text{Solve for } r. \\ \frac{33.75}{750} &= r \\ 0.045 &= r \end{aligned}$$

The annual interest rate is 4.5%.

(b) The year-end balances form an arithmetic sequence with $a = 783.75$ and $d = 33.75$.

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= 783.75 + (n - 1)33.75 \\ &= 783.75 + 33.75n - 33.75 \\ &= 750 + 33.75n \end{aligned}$$

(c) To find the balance in the account at the end of the 15th year, let $n = 15$.

$$\begin{aligned} t_n &= 750 + 33.75n \\ t_{15} &= 750 + 33.75(15) \\ &= 750 + 506.25 \\ &= 1256.25 \end{aligned}$$

OR

Find the interest earned over 15 years and add it to the principal.

$$\begin{aligned} I &= Prt \\ &= 750 \times 0.045 \times 15 \\ &= 506.25 \\ \text{amount} &= P + I \\ &= 750 + 506.25 \\ &= 1256.25 \end{aligned}$$

The balance in the account at the end of the 15th year is \$1256.25.

Example 5

Charlie opened a savings account as a child. He opened the account with only one deposit. At the end of the tenth year, there was \$300 in the account and, at the end of the 15th year, there was \$325 in the account. The account earns simple interest. Determine the annual interest rate and the original deposit.

Solution

Because the interest is simple, the year-end balances are the terms of an arithmetic sequence. The balance at the end of the tenth year is t_{10} and the balance at the end of the 15th year is t_{15}. So $t_{10} = 300$, $t_{15} = 325$, and $t_n = a + (n - 1)d$.

$$t_{15} = 325 \qquad\qquad t_{10} = 300$$
$$325 = a + (15 - 1)d \qquad\qquad 300 = a + (10 - 1)d$$
$$325 = a + 14d \qquad\qquad 300 = a + 9d$$

Solve the system of equations by elimination.

$$\begin{aligned} 325 &= a + 14d \quad ① \\ 300 &= a + 9d \quad ② \quad \text{Subtract.} \\ \hline 25 &= 5d \quad \text{Solve for } d. \\ 5 &= d \end{aligned}$$

To find a, let $d = 5$ in equation ①.

$$\begin{aligned} 325 &= a + 14d \\ 325 &= a + 14(5) \\ 325 &= a + 70 \\ 325 - 70 &= a \\ 255 &= a \end{aligned}$$

If $a = 255$, then the balance in the account at the end of the first year was \$255. The interest earned at the end of the first year was \$5. Then the original deposit was \$255 − \$5 = \$250.

To find the interest rate, substitute $P = \$250$, $I = \$5$, and $t = 1$ in the formula for simple interest.

$$\begin{aligned} I &= Prt \\ 5 &= 250 \times r \times 1 \\ \frac{5}{250} &= r \\ 0.02 &= r \end{aligned}$$

The annual interest rate is 2%.

Practise, Apply, Solve 1.6

A

1. **i.** Determine whether each sequence is arithmetic.
ii. If a sequence is arithmetic, then determine the rate of change between pairs of consecutive terms.

(a) 4, 11, 15, 19, … (b) 32, 23, 14, 5, … (c) 5, 6, 5, 6, …
(d) 2, 8, 32, 144, … (e) 22, 33, 44, 55, … (f) $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1, \ldots$

2. **i.** Determine whether each general term defines an arithmetic sequence.
ii. For each arithmetic sequence, determine the rate of change between pairs of consecutive terms.

(a) $t_n = 3n - 2$ (b) $t_n = 5^{n-1}$ (c) $t_n = -4n + 7$
(d) $t_n = (n + 1)(n - 1)$ (e) $t_n = \frac{1}{2}n + 2$ (f) $t_n = \frac{n + 2}{n}$

3. For each of the following arithmetic sequences, determine

i. the common difference **ii.** the general term **iii.** t_{10}

(a) 5, 10, 15, 20, … **(b)** $-30, -24, -18, -12, \ldots$

(c) 13, 11, 9, 7, … **(d)** $\frac{1}{3}, \frac{2}{3}, 1, 1\frac{1}{3}, \ldots$

(e) 0.2, 0.35, 0.5, 0.65, … **(f)** $-3, -3, -3, -3, \ldots$

4. Determine the recursive formula for each arithmetic sequence in question 3.

5. Find the general term for the arithmetic sequence in which

(a) the first term is 7 and the consecutive terms increase by 6

(b) the first term is -9 and the consecutive terms decrease by 2

(c) the first term is 25 and the consecutive terms decrease by 6

(d) the first term is 121 and the consecutive terms increase by 12

6. **Knowledge and Understanding:** An arithmetic sequence is $-5, 4, 13, 22, \ldots$.

(a) Find the next three terms.

(b) Find t_n.

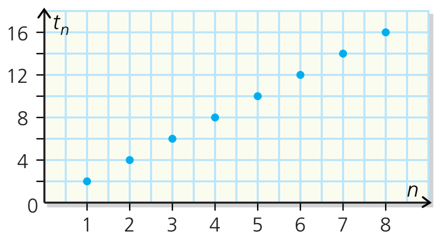
(c) What is t_{200}?

B

7. For each sequence, find

i. t_n, the general term **ii.** t_{24}

(a)



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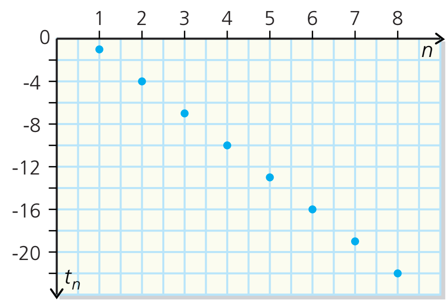

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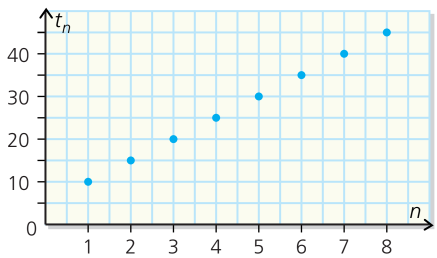

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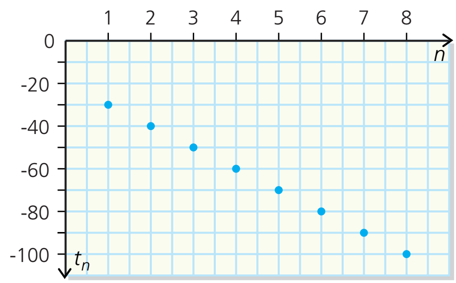


8. Graph each sequence.

(a) $t_n = 4n - 6$ (b) $t_n = -5n + 3$ (c) $t_n = -n - 1$

(d) $t_n = \frac{1}{2}n + 4$ (e) $t_n = \frac{n+1}{2}$ (f) $t_n = 10n - 2$

(g) $t_n = 3(n - 1)$ (h) $t_n = -2n + 4$

9. **Communication:**

(a) Graph the sequence defined by $t_n = -12 + 3n$.

(b) Use the graph to determine which term has a value of 21.

(c) Use the general term to determine which term has a value of 21.

(d) Compare the methods in (b) and (c). Which method gives the more reliable answer? Explain.

10. Calculate the simple interest in each situation.

(a) \$500 invested at 4%/a for 2 years

(b) \$3000 borrowed at 3.5%/a for 5 years

(c) \$4250 invested at 6%/a for 38 months

(d) \$5000 borrowed at 5.75%/a for 45 weeks

(e) \$900 invested at 8%/a for 60 days

11. Determine the number of terms in each arithmetic sequence.

(a) 8, 11, 14, 17, … , 71 (b) −10, −14, −18, −22, … , −138

(c) 6, 17, 28, 39, … , 435 (d) 21, 17, 13, 9, … , −35

(e) 8, 15, 22, 29, … , 99 (f) −92, −84, −76, −68, … , 52

12. Bob opens a savings account with a deposit of \$2000. The interest rate is 4.5%/a. He does not make any other deposits.

(a) By what amount will the year-end balance in the account increase each year?

(b) Write the sequence of the year-end balances for the first five years.

(c) Determine the general term of this sequence.

(d) Graph this sequence.

(e) At the end of one year, the balance in the account is \$3350. For how many years has the original deposit earned interest?

13. Carlo bought a house for \$175 000. His real estate agent told him that houses in the neighbourhood appreciate (increase) in value by \$5500 annually. When could he sell his house for at least twice the purchase price?

14. A pile of bricks has 85 bricks in the bottom row, 81 bricks in the second row, 77 bricks in the third row, and so on. There is only one brick in the top row.

(a) How many bricks are in the 13th row?

(b) How many rows are there in the pile?

15. **Application:** The most famous of all comets is Halley's comet, named for English astronomer Edmund Halley (1656–1742). He was the first to surmise that comets seen in 1531, 1607, and 1682 were, in fact, the same comet. He also correctly predicted the comet's return in 1758. The period of Halley's comet is about 76 years.

(a) Develop a formula to predict when Halley's comet will be visible from Earth.

(b) Predict when Halley's comet will be visible from Earth in our century.

(c) Predict when Halley's comet will be visible twice in a century in this millennium.

(d) Can you use this model to make precise predictions? Explain.

16. Write the recursive formula to generate the terms of any arithmetic sequence. Let a be the first term and d be the common difference.

17. For each of the following arithmetic sequences, determine a, d, and t_n.

(a) $t_6 = 23$, $t_{11} = 38$ (b) $t_8 = -49$, $t_{15} = -84$ (c) $t_{12} = 52$, $t_{16} = 80$

(d) $t_4 = 12$, $t_{21} = 20.5$ (e) $t_5 = 52$, $t_{17} = 124$ (f) $t_3 = 24$, $t_{19} = -88$

18. **Thinking, Inquiry, Problem Solving:** Suppose you earn 25¢ on September 1, 50¢ on September 2, 75¢ on September 3, and so on. How much money will you earn by the end of the month?

19. **Check Your Understanding**

(a) Determine t_n and t_{25} for the sequence 17, 23, 29, 35, … .

(b) List the properties that make a sequence arithmetic.

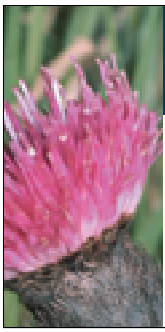
C

20. A water well-drilling company charges $40 for drilling the first metre, $44.50 for the second metre, $49 for the third metre, and so on. How much would the company charge to drill a 10-m well?

21. Show that the sequence $c, \frac{c + d}{2}, d$ is arithmetic.

22. Find the general term of the sequence $5x - 3, 3x - 4, x - 5, -x - 6, \ldots$.

23. How many multiples of 3 are between 40 and 1000?



The Chapter Problem—Controlling Non-Native Plant Populations

Apply what you learned to answer the following questions about the Chapter Problem on page 12.

CP11. Is the plant sequence you found in CP9 arithmetic? Explain.

CP12. Is the seed sequence you found in CP10 arithmetic? Explain.