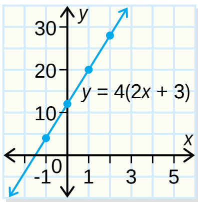
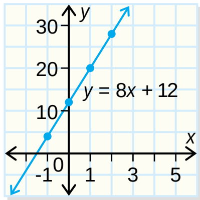
Trigonometric Identities

6.5 SKILL BUILDER

An equation that is true for all values of the variable in it is called an **identity**. For instance, the expression $4(2x + 3) = 8x + 12$ is an example of an algebraic identity because it is true for all values of x. Both sides of the expression are equivalent, or identical. If each side of the equation were separated and graphed, both graphs would be identical.

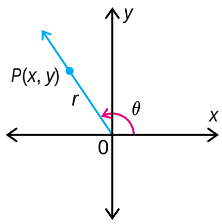




Some trigonometric equations can also be identities. However, it is not always obvious that both sides represent identical expressions. Showing that both sides of the equation represent the same expression proves that the original equation is an identity.

Fundamental Trigonometric Identities

Recall that for any angle, θ, in standard position, where $P(x, y)$ is a point on the terminal arm of the angle, the primary trigonometric ratios are



$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$

The Quotient Identity

Examining the ratio of $\frac{\sin\theta}{\cos\theta}$ allows you to develop an equivalent expression.

$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}}$$

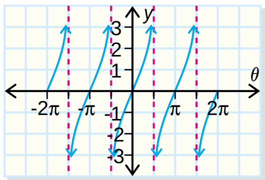
$$= \frac{y}{\cancel{r}_1} \times \frac{\cancel{r}^1}{x}$$

$$= \frac{y}{x}$$

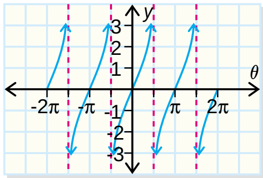
However, $\tan\theta = \frac{y}{x}$. Therefore,

$$\frac{\sin\theta}{\cos\theta} = \tan\theta \qquad ①$$

Equation ① is called a **quotient identity**. Both sides of this identity produce exactly the same graph.



$$y = \frac{\sin \theta}{\cos \theta}$$



$$y = \tan \theta$$

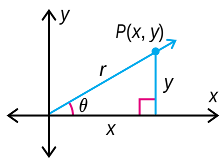
For any angle θ, the two expressions produce the exact same values, with the exception of where the functions are undefined. Therefore, the equation has an infinite number of solutions.

A Pythagorean Identity

For any angle θ in standard position, $x^2 + y^2 = r^2$.

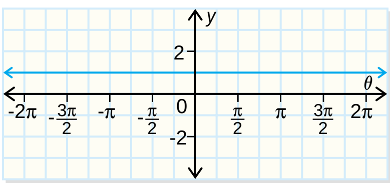
Because r is the distance from the origin to P, $r \neq 0$.

Divide the equation by r^2 to develop an equivalent expression.

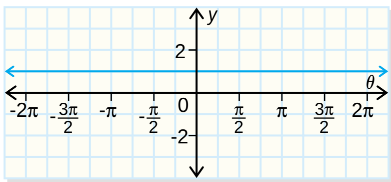


$$x^2 + y^2 = r^2$$
$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = \frac{r^2}{r^2}$$
$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$
$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$
$$(\sin \theta)^2 + (\cos \theta)^2 = 1$$
$$\sin^2 \theta + \cos^2 \theta = 1 \qquad ②$$

Equation ② is called a **Pythagorean identity**. Both sides of this identity produce exactly the same graph.



$$y = \sin^2 \theta + \cos^2 \theta$$



$$y = 1$$

For any angle, θ, the two expressions produce exactly the same value. Therefore, the equation has an infinite number of solutions. Rearranging this identity gives two other versions.

$$\sin^2 \theta = 1 - \cos^2 \theta \quad \text{and} \quad \cos^2 \theta = 1 - \sin^2 \theta$$

The quotient identity and this Pythagorean identity are often called the **fundamental trigonometric identities** because they can be used to prove that other more complicated equations are also identities. To prove that an equation is an identity, you must simplify the expression and show that the left side equals the right side.

It is often best to simplify the more complicated side first and rewrite expressions in terms of sine and cosine.

Example 1

Prove $\tan\theta\cos\theta = \sin\theta$.

Solution

L.S.	R.S.
$\tan\theta\cos\theta$	$\sin\theta$
$= \left(\frac{\sin\theta}{\cos\theta}\right)\cos\theta$	
$= \sin\theta$	

L.S. = R.S., therefore for all θ, $\tan\theta\cos\theta = \sin\theta$.

Example 2

Prove that $\tan^2\theta = \sin^2\theta\cos^{-2}\theta$.

Solution

L.S.	R.S.
$\tan^2\theta$	$\sin^2\theta\cos^{-2}\theta$
$= (\tan\theta)^2$	$= (\sin\theta)^2(\cos\theta)^{-2}$
$= \left(\frac{\sin\theta}{\cos\theta}\right)^2$	$= (\sin\theta)^2\left(\frac{1}{\cos\theta}\right)^2$
	$= \frac{(\sin\theta)^2}{(\cos\theta)^2}$
	$= \left(\frac{\sin\theta}{\cos\theta}\right)^2$

L.S. = R.S., therefore for all θ, $\tan^2\theta = \sin^2\theta\cos^{-2}\theta$.

Often, it is necessary to find a common denominator and then add or subtract expressions. In this case, always look for expressions involving $\sin^2\theta$ or $\cos^2\theta$. These are cases where the Pythagorean identity, $\sin^2\theta + \cos^2\theta = 1$, can be used. Sometimes it is necessary to factor, as Example 4 shows.

Example 3

Prove that $\tan x + \frac{1}{\tan x} = \frac{1}{\sin x\cos x}$.

Solution

L.S.	R.S.
$\tan x + \frac{1}{\tan x}$	$\frac{1}{\sin x\cos x}$
$= \frac{\sin x}{\cos x} + \frac{1}{\frac{\sin x}{\cos x}}$	$= \frac{1}{\cos x\sin x}$
$= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$	
$= \frac{\sin x\sin x + \cos x\cos x}{\cos x\sin x}$	
$= \frac{\sin^2 x + \cos^2 x}{\cos x\sin x}$	
$= \frac{1}{\cos x\sin x}$	

L.S. = R.S., therefore

$\tan x + \frac{1}{\tan x} = \frac{1}{\sin x\cos x}$

for all values of x.

Example 4

Prove that $\frac{\sin^2\theta}{1-\cos\theta} = 1 + \cos\theta$.

Solution

L.S.	R.S.
$\frac{\sin^2\theta}{1-\cos\theta}$	$1 + \cos\theta$
$= \frac{1-\cos^2\theta}{1-\cos\theta}$	
$= \frac{\overset{1}{\cancel{(1-\cos\theta)}}(1+\cos\theta)}{\underset{1}{\cancel{1-\cos\theta}}}$	
$= 1 + \cos\theta$	

L.S. = R.S., therefore,

$\frac{\sin^2\theta}{1-\cos\theta} = 1 + \cos\theta$ for all θ.

Key Ideas

- An **identity** is any mathematical equation that is true for all values of the variable. For example, $3(x + 1) = 3x + 3$ is an identity.
- A **trigonometric identity** is any mathematical equation with trigonometric expressions that is true for all values of the variable.

Fundamental Trigonometric Identities

Quotient Identity: $\frac{\sin x}{\cos x} = \tan x$

Pythagorean Identity: $\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$\cos^2 x = 1 - \sin^2 x$

- It is not always obvious that both sides of a trigonometric expression are equal. To prove that it is an identity, a proof that shows that both sides of the expression are equal is required.
- To prove that a given expression is an identity follow these steps.
  1. Separate the two sides of the expression.
  2. Simplify the more complicated side until it is identical to the other side or transform both sides of the expression into the same expression.
- The following strategies can be helpful for proving identities.
  1. Express all tangent functions in terms of the sine function or the cosine function.
  2. Look for expressions to which the Pythagorean identity can be applied.
  3. Where necessary, factor or find a common denominator.
- It is not always easy to prove an identity. If you get stuck or take a wrong turn, try another approach.

Practise, Apply, Solve 6.5

A

1. Use the definitions $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, and $\tan \theta = \frac{y}{x}$ to prove each identity.

(a) $\frac{\cos \theta}{\sin \theta} = \frac{1}{\tan \theta}$ (b) $1 + \tan^2 \theta = \frac{1}{\cos^2 \theta}$ (c) $1 - \sin^2 \theta = \cos^2 \theta$

2. Simplify.

(a) $\sin x\left(\frac{1}{\cos x}\right)$ (b) $(\cos x)(\tan x)$ (c) $1 - \cos^2 x$

(d) $1 - \sin^2 x$ (e) $\cos^2 x + \sin^2 x$ (f) $(1 - \sin x)(1 + \sin x)$

(g) $\frac{\tan x}{\sin x}$ (h) $\frac{\frac{\sin x}{\cos x}}{\tan x}$ (i) $\left(\frac{1}{\tan x}\right)\sin x$

(j) $\frac{1 + \tan^2 x}{\tan^2 x}$ (k) $\frac{\sin x \cos x}{1 - \sin^2 x}$ (l) $\frac{1 - \cos^2 x}{\sin x \cos x}$

(m) $\frac{1}{\sin x} + \frac{1}{\cos x}$ (n) $\tan x + \frac{1}{\cos x}$ (o) $\frac{1}{\tan x} + \sin x$

3. Factor each expression.

(a) $1 - \cos^2 \theta$ (b) $1 - \sin^2 \theta$

(c) $\sin^2 \theta - \cos^2 \theta$ (d) $\sin \theta - \sin^2 \theta$

(e) $\cos^2 \theta + 2 \cos \theta + 1$ (f) $\sin^2 \theta - 2 \sin \theta + 1$

4. (a) Prove that $\frac{\cos \alpha}{\tan \alpha} = \frac{1}{\sin \alpha} - \sin \alpha$, by expressing the left side in terms of $\sin \alpha$.

(b) Prove the identity.

5. (a) Prove that $\frac{\cos^2 x}{1 - \sin x} = 1 + \sin x$ by expressing $\cos^2 x$ in terms of $\sin x$.

(b) Factor the expression in (a).

(c) Prove the identity.

B

6. Verify each identity.

(a) $\frac{\sin x}{\tan x} = \cos x$

(b) $\frac{\tan \theta}{\cos \theta} = \frac{\sin \theta}{1 - \sin^2 \theta}$

(c) $\frac{1}{\cos \alpha} + \tan \alpha = \frac{1 + \sin \alpha}{\cos \alpha}$

(d) $1 - \cos^2 \theta = \tan \theta \cos \theta \sin \theta$

7. Prove each identity.

(a) $\frac{\sin^2 x}{1 - \cos x} = 1 + \cos x$

(b) $\frac{\tan^2 \alpha}{1 + \tan^2 \alpha} = \sin^2 \alpha$

(c) $\cos^2 x = (1 - \sin x)(1 + \sin x)$

(d) $\sin^2 \theta + 2 \cos^2 \theta - 1 = \cos^2 \theta$

8. Prove each identity.

(a) $(\sin x + \cos x)^2 = 1 + 2 \sin x \cos x$

(b) $\dfrac{1}{\sin^2 \theta} + \dfrac{1}{\cos^2 \theta} = \dfrac{1}{\sin^2 \theta - \sin^4 \theta}$

(c) $\cos x + \sin x \tan x = \dfrac{1}{\cos x}$

(d) $\dfrac{\sin^2 x}{\cos^2 x} \times \dfrac{1}{\tan x} = \tan x$

9. Prove each identity.

(a) $\dfrac{\cos^2 \theta - \sin^2 \theta}{\cos^2 \theta + \sin \theta \cos \theta} = 1 - \tan \theta$

(b) $\tan^2 x - \sin^2 x = \sin^2 x \tan^2 x$

(c) $\tan^2 x - \cos^2 x = \dfrac{1}{\cos^2 x} - 1 - \cos^2 x$

(d) $\dfrac{1}{1 + \cos \theta} + \dfrac{1}{1 - \cos \theta} = \dfrac{2}{\sin^2 \theta}$

10. Which equations are not identities? Justify your conclusion.

(a) $\dfrac{1 + 2 \sin \beta \cos \beta}{\cos \beta + \sin \beta} = \sin \beta + \cos \beta$

(b) $\dfrac{\tan x \sin x}{\tan x + \sin x} = \dfrac{\tan x \sin x}{\tan x \sin x}$

(c) $(1 - \cos^2 \theta)(1 - \tan^2 \theta) = \dfrac{\sin^2 \theta - 2 \sin^4 \theta}{1 - \sin^2 \theta}$

(d) $1 - 2 \cos^2 x = \sin^4 x - \cos^4 x$

11. Prove each identity.

(a) $\cos x \tan^3 x = \sin x \tan^2 x$

(b) $\sin^2 \theta + \cos^4 \theta = \cos^2 \theta + \sin^4 \theta$

(c) $(\sin x + \cos x)\left(\dfrac{\tan^2 x + 1}{\tan x}\right) = \dfrac{1}{\cos x} + \dfrac{1}{\sin x}$

(d) $\tan^2 \beta + \cos^2 \beta + \sin^2 \beta = \dfrac{1}{\cos^2 \beta}$

12. Prove that $\dfrac{\frac{1 - \cos x}{\cos x}}{\tan x} = \dfrac{\frac{\tan x}{1 + \cos x}}{\cos x}$.

13. **Check Your Understanding:** Prove that $\sin^2 x \left(1 + \dfrac{1}{\tan^2 x}\right) = 1$.

C

14. Prove each identity.

(a) $\dfrac{1 - \cos \beta}{\sin \beta} = \dfrac{\sin \beta}{1 + \cos \beta}$

(b) $\dfrac{\sin x}{1 + \cos x} = \dfrac{1}{\sin x} - \dfrac{1}{\tan x}$

15. Prove that $\dfrac{\sin^2 \phi + 2 \cos \phi - 1}{\sin^2 \phi + 3 \cos \phi - 3} = \dfrac{\cos^2 \phi + \cos \phi}{-\sin^2 \phi}$.

16. Prove that $\sin^2 \phi - \cos^2 \phi - \tan^2 \phi = \dfrac{2 \sin^2 \phi - 2 \sin^4 \phi - 1}{1 - \sin^2 \phi}$.