

Solving Trigonometry Problems in Two and Three Dimensions

6.2

Part 1: Problems in Two Dimensions

A new road requires building a bridge across a river. To determine the width of the river, a surveyor stakes out a base line of 300 m along the bank of a river. She places the line so that a tree on the opposite bank of the river lies between the ends of the line. From each end of the base line, she sights the position of the tree. The lines of sight to the tree make angles of 36° and 44° with the base line. Determine the width of the river.



Think, Do, Discuss

1. When solving any type of trigonometry problem, what will help you visualize the problem?
2. Draw a well-labelled sketch of the situation in the problem. Label all vertices and sides, and label all the given information.
3. For which kinds of triangles can you use the primary trigonometric ratios? Do you need to use these in this problem?
4. What information do you need to use the cosine law to solve a problem?
5. Do you need to use the cosine law in this problem?
6. What do you need to know to use the sine law to solve a problem?
7. Do you need to use the sine law in this problem?
8. List the steps to follow to solve this problem.
9. Carry out the steps. How wide is the river to the nearest tenth of a metre?
10. Will the river always be this wide? What other factors should the surveyor think about when building a bridge that can be used through all seasons? Explain.

A Problem-Solving Model

In mathematics, and other subjects, it is often useful to use a problem-solving model. Here is one possible model.

1. Understand the problem.

Draw a rough sketch. List the given information. List what must be found. List the mathematical concepts that could be used. List the steps that you think would lead to the solution.

2. Create a mathematical model of the problem.

Draw a well-labelled diagram. Mark all given information on it. Determine what information you need to find. Identify the relationships you will need to use.

3. Plan a solution.

Determine or explain how to use the relationships you identified in step 2.

4. Execute the plan.

Perform all the calculations in your plan.

5. Interpret and evaluate the solution.

Consider if your answer reasonable. Calculate the answer to the specified degree of accuracy.

6. Generalize your results.

Determine if your solution can be used in similar problems.

Part 2: Problems in Three Dimensions

While on a hiking trip in Algonquin Park, two hikers separate from their group and each becomes lost. Both hikers go to wide open clearings where aircraft can see them. After several hours, the larger group contacts the park rangers to tell them of the lost hikers. A rescue helicopter, at an altitude of 200 m, spots both hikers at the same time. One is at an angle of depression of 9° and a bearing of 240° . The other is at an angle of depression of 13° and a bearing of 68° . How far apart are they?

Think, Do, Discuss

Follow the problem-solving model.

1. Understand the problem.

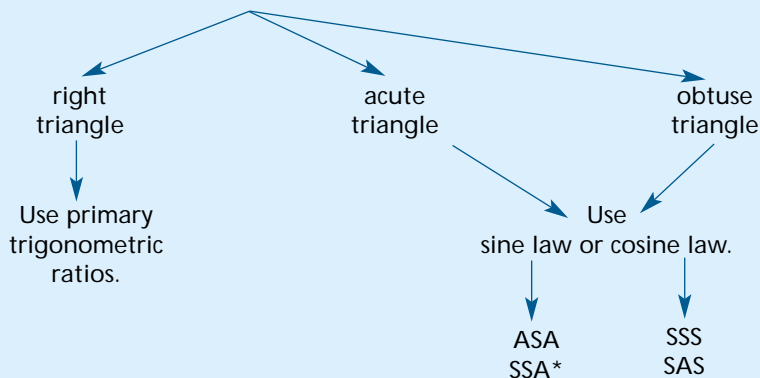
(a) Draw a diagram. Include two vertical right triangles that share the same height. The hypotenuse of each triangle represents the lines of sight from the helicopter to each hiker and the height of the triangles is the distance between the ground and the helicopter. Draw the horizontal triangle by joining the points that represent the hikers' positions. This line is the distance between the hikers.

- (b) List all the given information. Is any of it not needed?
 - (c) Write down what you need to know in order to find the distance between the lost hikers.
 - (d) List the mathematical concepts that might be useful for solving this problem. Explain how to use them.
 - (e) List the steps that you think would lead to the solution.
- 2. Create a mathematical model of the problem.**
- (a) Draw an accurate diagram. Label it so you can refer to the sides and angles by name.
 - (b) On your diagram, mark all the information that you have been given.
 - (c) What missing information do you need in order to determine the distance between the hikers? Will you use a vertical or horizontal triangle to find this distance?
 - (d) Identify the relationships that you need to use to find the missing information. Will you use vertical or horizontal triangles to find the missing information?
- 3. Plan a solution.**
- (a) What type of triangles must you use to determine the missing information? Which trigonometric ratios must be used?
 - (b) How can you find the angle between the two hikers?
 - (c) What type of triangle must you use to determine the distance between the two hikers?
 - (d) Do you need the sine law or cosine law in this problem?
- 4. Execute the plan.**
- (a) Determine the missing information, then determine the distance between the hikers.
- 5. Interpret and evaluate the solution.**
- (a) Describe how you can check whether your answer is reasonable.
 - (b) To what degree of accuracy should you calculate your answer?
- 6. Generalize your results.**
- (a) Can you always use this method to determine the distance between two objects when seen from above?
 - (b) Could you use this method to determine the distance between two elevated objects when seen from below?

Key Ideas

- When solving problems involving trigonometry, try to follow a problem-solving model. Consider the following:

Draw and label a diagram with all the given information.



*SSA is the ambiguous case and may have 0 sets, 1 set, or 2 sets of solutions.

- If both the sine law and cosine law can be used to solve a triangle, use the sine law, since it is the easier method.
- When solving a problem in three dimensions, it is helpful to separate the information into horizontal and vertical triangles.

Example 1

Mark is a landscaper who is creating a triangular planting garden. The homeowner wants the garden to have two equal sides and contain an angle of 135° . Also, the longest side of the garden must be exactly 5 m.

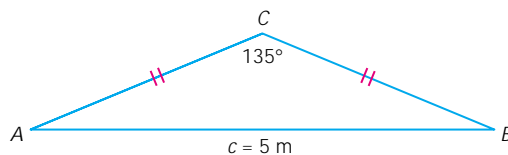
- How long is the plastic edging that Mark needs to surround the garden?
- Determine the area of the garden.

Solution

- Draw a well-labelled diagram.

Since two sides must be equal, the triangle is isosceles. Therefore, $AC = CB$.

$$\begin{aligned}
 \angle A &= \angle B \\
 &= (180^\circ - 135^\circ) \div 2 \\
 &= 22.5^\circ
 \end{aligned}$$



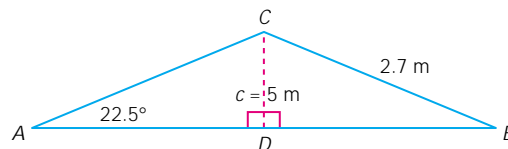
To determine the amount of edging needed, the perimeter of the triangle must be determined. Therefore, sides a and b must be determined. In this oblique triangle, two angles and a side are known (ASA). Therefore, use the sine law.

$$\begin{aligned}\frac{a}{\sin A} &= \frac{c}{\sin C} \\ \frac{a}{\sin 22.5^\circ} &= \frac{5}{\sin 135^\circ} \\ a &= \frac{5 \sin 22.5^\circ}{\sin 135^\circ} \\ a &= \frac{5(0.382\ 683\ 432\ 40)}{0.707\ 106\ 781\ 2} \\ a &\doteq 2.7\end{aligned}$$

$$\begin{aligned}\text{Perimeter} &= a + b + c \\ &= 2.7 + 2.7 + 5 \\ &= 10.4\end{aligned}$$

Mark will need about 10.4 m of plastic edging.

- (b) To determine the area of the garden, first determine the height of $\triangle ABC$. Draw CD , the height of the triangle, by drawing a line from vertex C to D , so $CD \perp AB$.



$$\begin{aligned}AC &= CB \\ &= 2.7\text{ m}\end{aligned}$$

$\triangle CAD$ is a right triangle, so use primary trigonometric ratios. Use the sine ratio to determine CD .

$$\begin{aligned}\sin 22.5^\circ &= \frac{CD}{2.7} \\ 2.7 (\sin 22.5^\circ) &= CD \\ 2.7(0.382\ 683\ 432\ 4) &= CD \\ 1.03 &\doteq CD\end{aligned}$$

$$\begin{aligned}\text{Area of } \triangle ABC &= \frac{1}{2}(bh) \\ &= \frac{1}{2}(AB)(CD) \\ &= \frac{1}{2}(5)(1.03) \\ &= 2.575\end{aligned}$$

The garden is 2.575 m² in area.

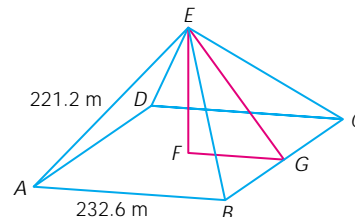
Example 2

The Great Pyramid at Giza in Egypt has a square base with sides of 232.6 m long. The distance from the top of the pyramid to each corner of the base was originally 221.2 m.

- (a) Determine the angle each face makes with the base.
 (b) Determine the size of the apex angle of a face of the pyramid.

Solution

- (a) Draw a well-labelled diagram. $\angle EGF$ is the angle each face makes with the base of the pyramid, where G is the midpoint of BC . To determine this angle, two sides of $\triangle EFG$ must be determined.



Quadrilateral $ABCD$ is a square. The height of the pyramid, EF , occurs at the exact centre of the base. Therefore,

$$\begin{aligned} FG &= \frac{AB}{2} \\ &= \frac{232.6}{2} \\ &= 116.3 \end{aligned}$$

In $\triangle EBG$, $EB = 221.2$ m. Also, since G is the midpoint of BC , then,

$$\begin{aligned} BG &= \frac{BC}{2} \\ &= \frac{232.6}{2} \\ &= 116.3 \end{aligned}$$

$\triangle EBC$ is isosceles and $EG \perp BC$. Therefore,

$$\begin{aligned} BG^2 + GE^2 &= EB^2 \\ 116.3^2 + GE^2 &= 221.2^2 \\ 13\,525.69 + GE^2 &= 48\,929.44 \\ GE^2 &= 48\,929.44 - 13\,525.69 \\ GE^2 &= 35\,403.75 \\ GE &= \sqrt{35\,403.75} \\ GE &\doteq 188.2 \end{aligned}$$

$EF \perp FG$ and $\triangle EGF$ is a right triangle. Therefore, use primary trigonometric ratios.

$$\begin{aligned} \cos(\angle EGF) &= \frac{FG}{GE} \\ \cos(\angle EGF) &= \frac{116.3}{188.2} \\ \cos(\angle EGF) &\doteq 0.617\,959\,617\,4 \\ \angle EGF &= \cos^{-1}(0.617\,959\,617\,4) \\ \angle EGF &\doteq 51.8^\circ \end{aligned}$$

- (b) The apex angle of each face is the vertical angle of each side of the pyramid. $\angle AEB$ is an apex angle. In $\triangle AEB$, all three sides are known (SSS). $AE = EB$ or 221.2 m and $AB = 232.6$ m. Therefore, use the cosine law.

$$\begin{aligned} AB^2 &= AE^2 + EB^2 - 2(AE)(EB) \cos(\angle AEB) && \text{Substitute.} \\ 232.6^2 &= 221.2^2 + 221.2^2 \\ &\quad - 2(221.2)(221.2) \cos(\angle AEB) && \text{Expand.} \\ 54\,102.76 &= 48\,929.44 + 48\,929.44 \\ &\quad - 97\,858.88 \cos(\angle AEB) && \text{Isolate } \cos(\angle AEB). \\ 97\,858.88 \cos(\angle AEB) &= 48\,929.44 + 48\,929.44 - 54\,102.76 \\ 97\,858.88 \cos(\angle AEB) &= 43\,756.12 \\ \cos(\angle AEB) &= \frac{43\,756.12}{97\,858.88} \\ \cos(\angle AEB) &= 0.447\,134\,894\,7 \end{aligned}$$

Determine $\angle AEB$.

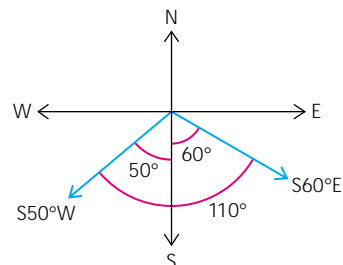
$$\begin{aligned}\cos^{-1}(0.447\ 134\ 894\ 7) &= \angle AEB \\ 63.4^\circ &= \angle AEB\end{aligned}$$

Example 3

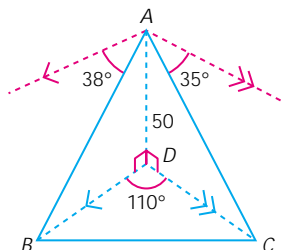
From the top of a 50-m high bridge, two boats are seen at anchor. One boat is $S50^\circ W$ and has an angle of depression of 38° . The other boat is $S60^\circ E$ and has a 35° angle of depression. How far apart are the boats?

Solution

To work in three dimensions, you need an accurate sketch to help visualize the situation. First visualize the bearings on the horizontal plane.



Now visualize the vertical planes as they intersect each bearing line.



To determine BC , the distance between the boats, use $\triangle BCD$. In this obtuse triangle, $\angle BDC$ and sides BD and CD are known (SAS). Therefore, use the cosine law. To determine BD , use right triangle ABD . To find CD , use right triangle ACD .

In $\triangle ABD$,

$$\begin{aligned}\angle BAD &= 90^\circ - 38^\circ \\ &= 52^\circ\end{aligned}$$

$$\text{Therefore, } \tan 52^\circ = \frac{BD}{50}$$

$$BD = 50 \tan 52^\circ$$

$$BD \doteq 64.0$$

In $\triangle ACD$,

$$\begin{aligned}\angle CAD &= 90^\circ - 35^\circ \\ &= 55^\circ\end{aligned}$$

$$\text{Therefore, } \tan 55^\circ = \frac{DC}{50}$$

$$DC = 50 \tan 55^\circ$$

$$DC \doteq 71.4$$

In $\triangle BCD$,

$$BC^2 = BD^2 + DC^2 - 2(BD)(DC) \cos D$$

$$BC^2 = 64^2 + 71.4^2 - 2(64)(71.4) \cos 110^\circ$$

$$BC^2 \doteq 4096 + 5097.96 - 9139.2(-0.342\ 020\ 143\ 3)$$

$$BC^2 = 12\ 319.750\ 49$$

$$\sqrt{BC^2} = \sqrt{12\ 319.750\ 49}$$

$$BC \doteq 110.99$$

The boats are about 111 m apart.

Practise, Apply, Solve 6.2

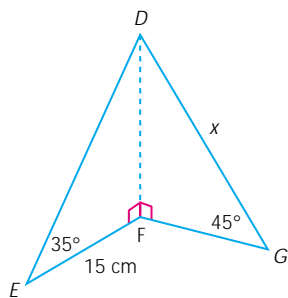
A

- Morana is trolling for salmon in Lake Ontario. She sets the fishing rod so its tip is 1 m above the water and the line enters the water at an angle of 35° . Fish have been tracked at a depth of 45 m. What length of line must she let out?
- Josh is building a garden shed that is 4 m wide. The two sides of the roof must meet at an 80° angle and be equal in length. How long must each rafter be if he allows for a 0.5-m overhang?
- A parallelogram has sides of 12 cm and 15 cm. The contained angle is 75° . How long is
 - the shorter diagonal?
 - the longer diagonal?
- The height of any isosceles triangle begins from the midpoint of the base of the triangle. Suppose $\triangle ABC$ is isosceles, with the two equal sides being 10 cm and the equal angles being 40° . Determine the height and area of the triangle.
- A regular hexagon is inscribed inside a circle with radius 10 cm. Determine the perimeter of the hexagon.
- The blueprints for the roof of a new house call for one side to have a slope of $\frac{1}{3}$ and the other side to have a slope of $\frac{1}{2}$. Determine the measure of the angle at the peak of the roof.

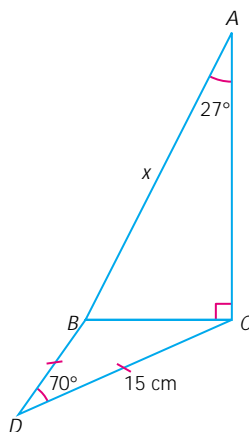
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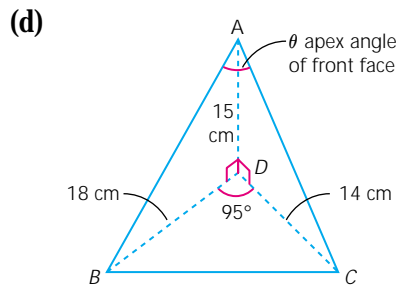
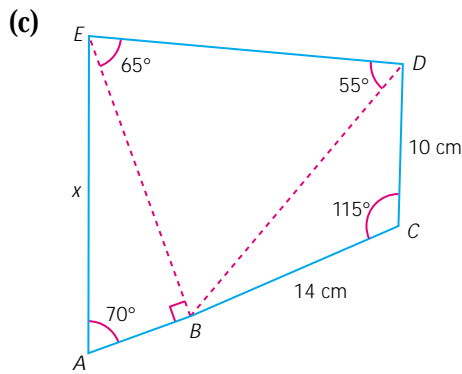
- Determine the measure of the indicated side or angle correct to the accuracy given in the question.

(a)

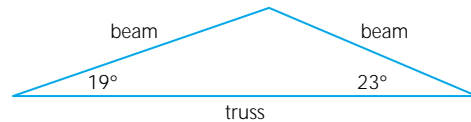


(b)





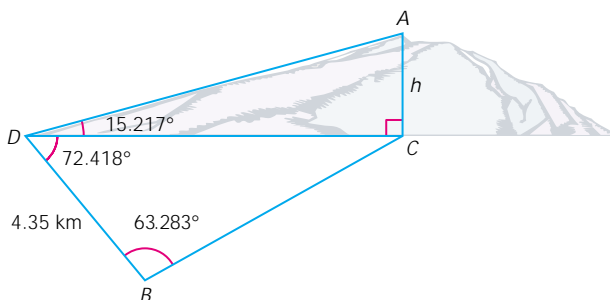
8. **Communication:** Think of a real-life problem that can be modelled by a three-dimensional diagram in which a side or distance must be determined. Describe the problem, sketch the situation, and explain what must be done to solve the problem.
9. **Knowledge and Understanding:** A roof truss is 10.4 m wide and the angles formed by the roof beams are 19° and 23° . How long is each roof beam?



10. The Leaning Tower of Pisa leans toward the south at an angle of about 5.5° . On one day, its shadow is 90 m long, and the angle of elevation from the tip of the shadow to the top of the tower is 32° .
- What is the slant height of the tower?
 - How high is the tip of the tower above the ground?
11. Two ships, the *Argus* and the *Baffin*, are 125.0 nautical miles apart. They both hear a homing signal from a lifeboat. The captain of the *Argus* calculates that the angle formed between the *Baffin*, itself, and the lifeboat is 36° . The captain of the *Baffin* calculates the angle between the *Argus*, itself, and the lifeboat is 42° .
- How far is each ship from the lifeboat?
 - The *Argus* has a top speed of 9.7 nautical miles per hour. The top speed of the *Baffin* is 9.4 nautical miles per hour. Which ship should reach the lifeboat first?
12. A surveyor needs to estimate the length of a swampy area. She starts at one end of the swamp and walks in a straight line, 450 paces and turns 60° towards the swamp. She then walks in another straight line, 380 paces before arriving at the other end of the swamp. One pace is about 75 cm. Estimate the length of the swamp in metres.



13. Two planes, *Able* and *Baker*, are 31 km apart when plane *Able* is 38 km from an airstrip. The angle between the two planes, as measured from the airstrip, is 46° . Both planes are flying at the same speed. Which plane will land first? Explain.
14. **Application:** Two forest fire towers, *A* and *B*, are 20.3 km apart. The bearing from *A* to *B* is $N70^\circ E$. The ranger in each tower observes a fire and radios the fire's bearing from the tower. The bearing from tower *A* is $N25^\circ E$. From tower *B*, the bearing is $N15^\circ W$. How far is the fire from each tower?
15. Two roads intersect at an angle of 48° . A car and a truck collide at the intersection, and then leave the scene of the accident on different roads. The car travels at 100 km/h, while the truck goes 80 km/h. Fifteen minutes after the accident, a police helicopter locates the car and pulls it over. Twenty minutes after the accident, a police cruiser pulls over the truck. How far apart are the car and the truck at this time?
16. A surveyor uses a diagram to help determine the height, h , of a mountain.

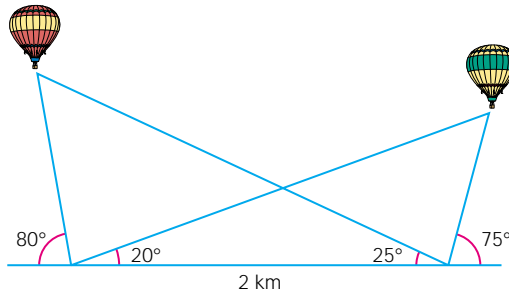


- (a) Use $\triangle BDC$ to determine $\angle C$.
- (b) Use $\triangle BDC$ and the sine law to determine DC .
- (c) Use $\triangle ADC$ to calculate h .
17. Two roads intersect at an angle of 12° . Two cars leave the intersection, each on a different road. One car travels at 90 km/h and the other car at 120 km/h. After 20 min, a police helicopter 1000 m directly above and between the cars, notes the angle of depression of the slower car is 14° . What is the horizontal distance from the helicopter to the faster car?
18. **Thinking, Inquiry, Problem Solving:** A given pyramid has a regular hexagonal base. Each side of the base is 12.5 cm and the vertical height of the pyramid is 20.0 cm. Determine
- (a) the measure of the apex angle of each face
- (b) the surface area of the pyramid
- (c) the volume of the pyramid

19. **Check Your Understanding:** When you are given a problem to solve that involves trigonometry, what essential thing must you do to solve each problem? Explain how you would decide whether to use the primary trigonometric ratios, the sine law, the cosine law, or a combination of these.

C

20. Two hot air balloons are moored directly over a level road. The diagram shows the angle of elevation of the balloons from two observers 2 km apart.
- (a) To the nearest tenth of a kilometre, how far apart are the balloons?
- (b) Which balloon is higher, and by how many metres?



The Chapter Problem — What Time Is It?

Apply what you have learned to answer these questions about the Chapter Problem on page 494.

- CP4. Assume the pendulum is 80 cm long. How far must it be pulled back from the centre or rest position (horizontal distance), to swing through an angle of $\frac{5\pi}{8}$?
- CP5. How long will this pendulum take to complete one swing? one period?
- CP6. Does the pendulum need to be longer or shorter than 80 cm for one swing to occur in 1 s? Explain your reasoning.