EXERCISES 10.2.1

1. Prove the identity

(a) $\sin\theta + \cot\theta\cos\theta = \operatorname{cosec}\theta$

(b) $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} = \dfrac{2}{\sin\theta}$

(c) $\dfrac{\sin^2\theta}{1-\cos\theta} = 1+\cos\theta$

(d) $3\cos^2 x - 2 = 1 - 3\sin^2 x$

(e) $\tan^2 x\cos^2 x + \cot^2 x\sin^2 x = 1$

(f) $\sec\theta - \sec\theta\sin^2\theta = \cos\theta$

(g) $\sin^2\theta(1+\cot^2\theta) - 1 = 0$

(h) $\dfrac{1}{1-\sin\phi} + \dfrac{1}{1+\sin\phi} = 2\sec^2\phi$

(i) $\dfrac{\cos\theta}{1+\sin\theta} + \tan\theta = \sec\theta$

(j) $\dfrac{1-\sin\theta}{\cos\theta} = \dfrac{\cos\theta}{1+\sin\theta}$

(k) $\dfrac{1}{\sec x + \tan x} = \sec x - \tan x$

(l) $\sin x + \dfrac{\cos^2 x}{1+\sin x} = 1$

(m) $\dfrac{\sec\phi + \operatorname{cosec}\phi}{\tan\phi + \cot\phi} = \sin\phi + \cos\phi$

(n) $\dfrac{\sin x + 1}{\cos x} = \dfrac{\sin x - \cos x + 1}{\sin x + \cos x - 1}$

(o) $\tan x + \sec x = \dfrac{\tan x + \sec x - 1}{\tan x - \sec x + 1}$

2. Prove the following

(a) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$

(b) $\sec^2\theta\operatorname{cosec}^2\theta = \sec^2\theta + \operatorname{cosec}^2\theta$

(c) $\sin^4 x - \cos^4 x = (\sin x + \cos x)(\sin x - \cos x)$

(d) $\sec^4 x - \sec^2 x = \tan^4 x + \tan^2 x$

(e) $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \sin x\cos x$

(f) $(\cot x - \operatorname{cosec} x)^2 = \dfrac{\sec x - 1}{\sec x + 1}$

(g) $(2b\sin x\cos x)^2 + b^2(\cos^2 x - \sin^2 x)^2 = b^2$

3. Eliminate θ from each of the following pairs

(a) $x = k\sin\theta,\ y = k\cos\theta$

(b) $x = b\sin\theta,\ y = a\cos\theta$

(c) $x = 1+\sin\theta,\ y = 2-\cos\theta$

(d) $x = 1-b\sin\theta,\ y = 2+a\cos\theta$

(e) $x = \sin\theta + 2\cos\theta,\ y = \sin\theta - 2\cos\theta$

4. (a) If $\tan\theta = \dfrac{3}{4},\ \pi \le \theta \le \dfrac{3\pi}{2}$, find i. $\cos\theta$ ii. $\operatorname{cosec}\theta$

(b) If $\sin\theta = -\dfrac{3}{4},\ \dfrac{3\pi}{2} \le \theta \le 2\pi$, find i. $\sec\theta$ ii. $\cot\theta$

5. Solve the following, where $0 \le \theta \le 2\pi$

(a) $4\sin\theta = 3\operatorname{cosec}\theta$

(b) $2\cos^2\theta + \sin\theta - 1 = 0$

(c) $2 - \sin\theta = 2\cos^2\theta$

(d) $2\sin^2\theta = 2 + 3\cos\theta$

6. Prove $\sin^2x(1+n\cot^2x)+\cos^2x(1+n\tan^2x) = \sin^2x(n+\cot^2x)+\cos^2x(n+\tan^2x)$.

7. If $k\sec\phi = m\tan\phi$, prove that $\sec\phi\tan\phi = \frac{mk}{m^2-k^2}$.

8. If $x = k\sec^2\phi + m\tan^2\phi$ and $y = l\sec^2\phi + n\tan^2\phi$, prove that $\frac{x-k}{k+m} = \frac{y-l}{l+n}$.

9. Given that $\tan\theta = \frac{2a}{a^2-1}, 0<\theta<\frac{\pi}{2}$, find (a) $\sin\theta$ (b) $\cos\theta$

10. (a) If $\sin x+\cos x = 1$, find the values of

i. $\sin^3x+\cos^3x$ ii. $\sin^4x+\cos^4x$

(b) Hence, deduce the value of $\sin^kx+\cos^kx$, where k is a positive integer.

11. If $\tan\phi = -\frac{1}{\sqrt{x^2-1}}, \frac{\pi}{2}<\phi<\pi$, find, in terms of x,

(a) $\sin\phi+\cos\phi$ (b) $\sin\phi-\cos\phi$ (c) $\sin^4\phi-\cos^4\phi$

12. Find (a) the maximum value of (b) the minimum value of

i. $\cos^2\theta+5$ ii. $\frac{5}{3\sin^2\theta+2}$ iii. $2\cos^2\theta+\sin\theta-1$

13. (a) Given that $b\sin\phi = 1$ and $b\cos\phi = \sqrt{3}$, find b.

(b) Hence, find all values of ϕ that satisfy the relationship described in (a).

14. Find (a) the maximum value of (b) the minimum value of

i. $5^{3\sin\theta-1}$ ii. $3^{1-2\cos\theta}$

15. Given that $\sin\theta\cos\theta = k$, find (a) $(\sin\theta+\cos\theta)^2$, $\sin\theta+\cos\theta>0$.

(b) $\sin^3\theta+\cos^3\theta$, $\sin\theta+\cos\theta>0$.

16. (a) Given that $\sin\phi = \frac{1-a}{1+a}, 0<\phi<\frac{\pi}{2}$, find $\tan\phi$.

(b) Given that $\sin\phi = 1-a, \frac{\pi}{2}<\phi<\pi$, find i. $2-\cos\phi$

ii. $\cot\phi$

17. Find

(a) the value(s) of $\cos x$, where $\cot x = 4(\operatorname{cosec}x-\tan x)$, $0<x<\pi$.

(b) the values of $\sin x$, where $3\cos x = 2+\frac{1}{\cos x}, 0\le x\le 2\pi$.

18. Given that $\sin 2x = 2\sin x\cos x$, find all values of x, such that $2\sin 2x = \tan x, 0\le x\le\pi$.

To prove a given identity, any one of the following approaches can be used:

1. Start with the L.H.S and then show that L.H.S = R.H.S
2. Start with the R.H.S and then show that R.H.S = L.H.S
3. Show that L.H.S = p, show that R.H.S = p $\Rightarrow$ L.H.S = R.H.S
4. Start with L.H.S = R.H.S $\Rightarrow$ L.H.S – R.H.S = 0 .

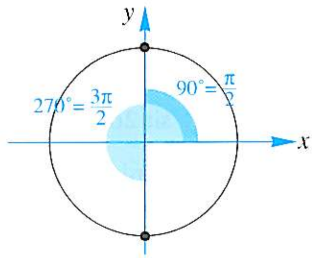
When using approaches 1., and 2., choose whichever side has more to work with.

EXAMPLE 10.15

Find all values of x, such that $\sin 2x = \cos x$, where $0 \le x \le 2\pi$.

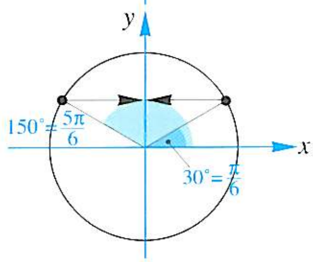
Solution

$$\sin 2x = \cos x \Leftrightarrow 2\sin x\cos x = \cos x$$
$$\Leftrightarrow 2\sin x\cos x - \cos x = 0$$
$$\Leftrightarrow \cos x(2\sin x - 1) = 0$$
$$\Leftrightarrow \cos x = 0 \text{ or } \sin x = \frac{1}{2}$$



Now, $\cos x = 0, 0 \le x \le 2\pi \Leftrightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$

and $\sin x = \frac{1}{2}, 0 \le x \le 2\pi \Leftrightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$



EXERCISES 10.2.2

1. Given that $\sin\theta = -\frac{5}{6}, \frac{3\pi}{2} \le \theta \le 2\pi$, evaluate

   (a) $\sin 2\theta$ (b) $\cos 2\theta$ (c) $\tan 2\theta$ (d) $\sin 4\theta$

2. Given that $\tan x = -3, \frac{\pi}{2} \le \theta \le \pi$, evaluate

   (a) $\sin 2x$ (b) $\cos 2x$ (c) $\tan 2x$ (d) $\tan 4x$

3. Find the exact value of $\sin\frac{\pi}{12}$

4. Given that $\tan x = \frac{a}{b}, \pi \le \theta \le \frac{3\pi}{2}$, evaluate

   (a) $\sin 2x$ (b) $\operatorname{cosec} 2x$ (c) $\cos 4x$ (d) $\tan 2x$

5. Prove the following identities:

(a) $\tan(\theta + \phi) + \tan(\theta - \phi) = \dfrac{2\sin 2\theta}{\cos 2\theta + \cos 2\phi}$

(b) $\dfrac{1 + \cos 2y}{\sin 2y} = \dfrac{\sin 2y}{1 - \cos 2y}$

(c) $\cos^4\alpha - \sin^4\alpha = 1 - 2\sin^2\alpha$

(d) $\dfrac{1}{\sin y \cos y} - \dfrac{\cos y}{\sin y} = \tan y$

(e) $\dfrac{1 + \sin 2\theta}{\cos 2\theta} = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$

(f) $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

(g) $\cos\beta + \sin\beta = \dfrac{\cos 2\beta}{\cos\beta - \sin\beta}$

(h) $\sin^2\dfrac{\theta}{2} = \dfrac{1 - \cos\theta}{2}$

(i) $\dfrac{\sin^3 x + \cos^3 x}{\sin x + \cos x} = 1 - \dfrac{1}{2}\sin 2x$

(j) $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

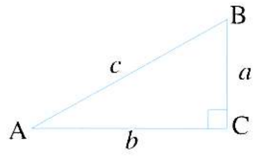
6. For the right-angled triangle shown, prove that

(a) $\sin 2\alpha = \dfrac{2ab}{c^2}$

(b) $\cos 2\alpha = \dfrac{b^2 - a^2}{c^2}$

(c) $\sin\dfrac{1}{2}\alpha = \sqrt{\dfrac{c - b}{2c}}$

(d) $\cos\dfrac{1}{2}\alpha = \sqrt{\dfrac{c + b}{2c}}$



7. Solve the following for $0 \le x \le 2\pi$

(a) $\sin x = \sin 2x$

(b) $\sin x = \cos 2x$

(c) $\tan 2x = 4\tan x$