# Lesson 38 - Laws of Logarithms

Math 2 Honors - Santowski



## Lesson Objectives

- Understand the rationale behind the "properties of logs"
- Apply the various properties of logarithms in solving equations and simplifying expressions



## Investigation ➔ Number Patterns

- Use your calculator to add log(5) + log(6)
- What is the answer?
- What does the answer represent?
- What is the base?
- So that would suggest …..??
- Which, given the original question, suggests that …..?



## Investigation ➔ Number Patterns

- Use your calculator to add log(5) + log(6)
- What is the answer? ➔ 1.477121255
- What does the answer represent? ➔ an exponent
- What is the base? ➔ 10
- So that would suggest …..?? ➔ $10^{(1.477121225)} = 30$
- Which, given the original question, suggests that …..? ➔ log(5) + log(6) = log(30)



## Investigation ➔ Number Patterns

- Try the same "number patterns" investigation with:
- (a) log(12) + log(5) =
- (b) log(125) + log(8) =
- (c) log (¼) + log(200) =
- So a property of logs that is being suggested is ….. ????



## (A) Properties of Logarithms – Product Law

- Recall the laws for exponents ➔product of powers ➔ $(b^x)(b^y) = b^{(x+y)}$ ➔ so we ADD the exponents when we MULTIPLY powers
- For example ➔ $(2^3)(2^5) = 2^{(3+5)}$
- So we have our POWERS ➔ 8 x 32 = 256

## (A) Properties of Logarithms – Product Law

- Now, let's consider this from the INVERSE viewpoint
- We have the ADDITION of the exponents ➔ 3 + 5 = 8
- But recall from our work with logarithms, that the exponents are the OUTPUT of logarithmic functions
- So ➔ 3 + 5 = 8 becomes $\log_2 8 + \log_2 32 = \log_2 256$
- Now, HOW do we get the right side of our equation to equal the left?
- Recall that 8 x 32 = 256
- So $\log_2(8 \times 32) = \log_2 8 + \log_2 32 = \log_2 256$



## (A) Properties of Logarithms – Product Law

- So we have our first law ➔ when adding two logarithms, we can simply write this as a single logarithm of the product of the 2 powers
- $\log_a(mn) = \log_a m + \log_a n$
- $\log_a m + \log_a n = \log_a(mn)$



## Investigation ➔ Number Patterns

- Use your calculator to add log(56) - log(7)
- What is the answer?
- What does the answer represent?
- What is the base?
- So that would suggest …..??
- Which, given the original question, suggests that …..?



## Investigation ➔ Number Patterns

- Use your calculator to add log(56) - log(7)
- What is the answer? ➔ 0.903089987
- What does the answer represent? ➔ an exponent
- What is the base? ➔ 10
- So that would suggest …..?? ➔ $10^{(0.903089987)} = 8$
- Which, given the original question, suggests that …..? ➔ log(56) - log(7) = log(8)



## Investigation ➔ Number Patterns

- Try the same "number patterns" investigation with:
- (a) log(12) - log(4) =
- (b) log(125) - log(5) =
- (c) log (12000) - log(200) =
- So a property of logs that is being suggested is ….. ????



## (B) Properties of Logarithms – Quotient Law

- Recall the laws for exponents ➔Quotient of powers ➔ $(b^x)/(b^y) = b^{(x-y)}$ ➔ so we SUBTRACT the exponents when we DIVIDE powers
- For example ➔ $(2^8)/(2^3) = 2^{(8-3)}$
- So we have our POWERS ➔ 256 ÷ 8 = 32

## (B) Properties of Logarithms – Quotient Law

- Now, let's consider this from the INVERSE viewpoint
- We have the SUBTRACTION of the exponents ➔ 8 - 3 = 5
- But recall from our work with logarithms, that the exponents are the OUTPUT of logarithmic functions
- So ➔ 8 - 3 = 5 becomes $\log_2 256 - \log_2 8 = \log_2 32$
- Now, HOW do we get the right side of our equation to equal the left?
- Recall that 256/8 = 32
- So $\log_2(256/8) = \log_2 256 - \log_2 8 = \log_2 32$



## (B) Properties of Logarithms – Quotient Law

- So we have our second law ➔ when subtracting two logarithms, we can simply write this as a single logarithm of the quotient of the 2 powers
- $\log_a(m/n) = \log_a m - \log_a n$
- $\log_a m - \log_a n = \log_a(m/n)$



## (C) Properties of Logarithms- Logarithms of Powers

- Now work with $\log_3(625)$
- we can rewrite as: ……



## (C) Properties of Logarithms- Logarithms of Powers

- Now work with $\log_3(625) = \log_3(5^4) = x$ :
- we can rewrite as $\log_3(5 \times 5 \times 5 \times 5) = x$
- we can rewrite as $\log_3(5) + \log_3(5) + \log_3(5) + \log_3(5) = x$
- We can rewrite as $4 [\log_3(5)]$
- So we can generalize as $\log_3(5^4) = 4 [\log_3(5)]$
- So if $\log_3(625) = \log_3(5)^4 = 4 \times \log_3(5)$ ➔ It would suggest a rule of logarithms ➔ $\log_a(b^x) = x \log_a b$



## (D) Properties of Logarithms – Logs as Exponents

- Simplify $3^{\log_3 5} = x$



## (D) Properties of Logarithms – Logs as Exponents

- Consider the example $3^{\log_3 5} = x$
- Recall that the expression $\log_3(5)$ simply means "the exponent on 3 that gives 5" ➔ let's call that *y*
- So we are then asking you to place that same exponent (the *y*) on the same base of 3
- Therefore taking the exponent that gave us 5 on the base of 3 (*y*) onto a 3 again, must give us the same 5!!!!
- We can demonstrate this algebraically as well

## (D) Properties of Logarithms – Logs as Exponents

- Let's take our exponential equation and write it in logarithmic form
- So $3^{\log_3 5} = x$ becomes $\log_3(x) = \log_3(5)$
- Since both sides of our equation have a $\log_3$ then the power *x* equals the power 5 as we had tried to reason out in the previous slide
- So we can generalize that $b^{\log_b x} = x$



## (E) Summary of Laws

| | |
|---|---|
| Product Rule | $\log_a(mn) = \log_a m + \log_a n$<br>$\log_a m + \log_a n = \log_a(mn)$ |
| Quotient Rule | $\log_a(m/n) = \log_a(m) - \log_a(n)$<br>$\log_a(m) - \log_a(n) = \log_a(m/n)$ |
| Power Rule | $\log_a(m^p) = (p) \times (\log_a m)$<br>$(p) \times (\log_a m) = \log_a(m^p)$ |
| Logs as exponents | $b^{\log_b x} = x$ |



## (F) Examples

- (i) $\log_3 54 + \log_3(3/2)$
- (ii) $\log_2 144 - \log_2 9$
- (iii) $\log 30 + \log(10/3)$
- (iv) which has a greater value
  - (a) $\log_3 72 - \log_3 8$ or (b) $\log 500 + \log 2$
- (v) express as a single value
  - (a) $3\log_2 x + 2\log_2 y - 4\log_2 a$
  - (b) $\log_3(x+y) + \log_3(x-y) - (\log_3 x + \log_3 y)$
- (vi) $\log_2(3/4) - \log_2(24)$
- (vii) $(\log_2 5 + \log_2 25.6) - (\log_2 16 + \log_3 9)$



## (F) Examples

- Solve for x

$$\log_2 x = 2\log_2 7 + \log_2 3$$
$$\log_2 x + \log_2 11 = \log_2 \sqrt{99}$$
$$\log \sqrt[3]{x} + \log 13 = -\log \frac{1}{91}$$

- Solve for x and verify your solution

$$\log_5(x+1) + \log_5 3 = 2$$
$$\log_3(x-2) + \log_3 x = 1$$
$$\log x + \log(x-5) = \log(2x-12)$$



## (F) Examples

- Solve and verify

$$\log_2 x = \frac{1}{3}\log_2 3 + \log_2 \sqrt{3}$$
$$\log_5(x-1) - \log_5(x-5) = \log_5 \frac{1}{x+3}$$
$$\log 250 - \log 2 = 3\log \frac{1}{x}$$

- If $a^2 + b^2 = 23ab$, prove that

$$\log\left(\frac{a+b}{5}\right) = \frac{\log a + \log b}{2}$$



## (F) Examples

- (1) Evaluate $\log_2\left[8 \times \sqrt{32}\right] + \log_7\left[49 \times \sqrt[4]{7}\right]$
- (2) Prove that a = 1/6 given that

$$\log_m p = 2a^2 \quad \text{and} \quad a\log_p m = 3$$

- (3) If $a^2 + b^2 = 23ab$, prove that

$$\log\left(\frac{a+b}{5}\right) = \frac{\log a + \log b}{2}$$

## (F) Examples

Use the logarithm laws to simplify the following:

(a) $\log_2 xy - \log_2 x^2$

(b) $\log_2 \frac{8x^2}{y} + \log_2 2xy$

(c) $\log_3 9xy^2 - \log_3 27xy$

(d) $\log_4 (xy)^3 - \log_4 xy$

(e) $\log_3 9x^4 - \log_3 (3x)^2$



## (F) Examples

- Write each single log expression as a sum/difference/product of logs

$$\log \frac{abc^2}{d^3}$$

$$\log \frac{x^3\sqrt{y}}{z^5}$$

$$\ln \sqrt{\sin x \ln x}$$



## (F) Examples

**Exercise 2.** *Given* $\text{Log}_{10}(5.0) = 0.70$ $\text{Log}_{10}(2.0) = 0.30$ $\text{Log}_{10}(3.0) = 0.48$, without a calculator, determine:

(1) $\text{Log}_{10}(6.0)$

(2) $\text{Log}_{10}(8.0)$

(3) $\text{Log}_{10}(\frac{1}{2})$

(4) $\text{Log}_{10}(15.)$

(5) $\text{Log}_{10}(\frac{2}{3})$

(6) $\text{Log}_{10}(0.40)$

(7) $\text{Log}_{10}(\frac{4}{15})$

(8) $\text{Log}_{10}(\sqrt{5.0})$

(9) $\text{Log}_{10}(\sqrt[4]{3.0})$

(10) $\text{Log}_{10}(0.036)$



## (F) Examples

- (a) Predict the appearance of the graph. Justify your reasoning.

$$f(x) = 10^{\log x}$$

- (b) Predict the appearance of the graph. Justify your reasoning.

$$f(x) = \log\left(10^x\right)$$

- (c) Predict the appearance of the graph. Justify your reasoning.

$$f(x) = \log_b \left(\frac{1}{x}\right)$$



## (F) Examples

- (a) If f(x) = $\log_2(x)$, show that f(2x) = f(x) + 1
- (b) What 2 transformations are thus identical?
- (c) If f(x) = $\log_3(x)$, show that f(x/3) = f(x) – 1
- (d) What 2 transformations are thus identical?
- (e) What pattern can you see? Predict a "general statement."
- (f) Prove your general statement.



## (G) Internet Links

- Logarithm Rules Lesson from Purple Math
- College Algebra Tutorial on Logarithmic Properties from West Texas AM

## (H) Homework

- p. 382 # 14, 16, 21, 24, 28, 33, 35, 37, 39, 42, 48, 54, 57, 59, 61, 67, 69, 71-73