Lesson 23 – Roots of Polynomial Functions

Math 2 Honors - Santowski

Lesson Objectives

- Reinforce the understanding of the connection between factors and roots
- Mastery of the factoring of polynomials using the algebraic processes of long & synthetic division & various theorems like RRT, RT & FT
- Introduce the term "multiplicity of roots" and illustrate its graphic significance
- Solve polynomial equations for x being an element of the set of complex numbers
- State the Fundamental Theorem of Algebra

(A) Multiplicity of Roots

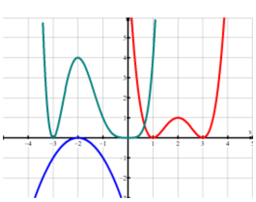
- Factor the following polynomials:
- $P(x) = x^2 – 2x – 15$
- $P(x) = x^2 – 14x + 49$
- $P(x) = x^3 + 3x^2 + 3x + 1$
- Now solve each polynomial equation, P(x) = 0
- Solve $0 = 5(x + 1)^2(x – 2)^3$
- Solve $0 = x^4(x – 3)^2(x + 5)$
- Solve $0 = (x + 1)^3(x – 1)^2(x – 5)(x + 4)$

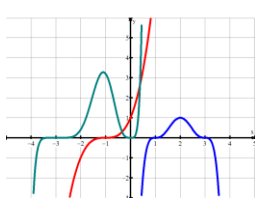
(A) Multiplicity of Roots

- If *r* is a zero of a polynomial and the exponent on the factor that produced the root is *k*, $(x – r)^k$, then we say that *r* has **multiplicity** of *k*. Zeroes with a multiplicity of 1 are often called **simple** zeroes.
- For example, the polynomial $x^2 – 14x + 49$ will have one zero, x = 7, and its multiplicity is 2. In some way we can think of this zero as occurring twice in the list of all zeroes since we could write the polynomial as, $(x – 7)^2 = (x – 7)(x – 7)$
- Written this way the term (x – 7) shows up twice and each term gives the same zero, x = 7.
- Saying that the multiplicity of a zero is *k* is just a shorthand to acknowledge that the zero will occur *k* times in the list of all zeroes.

(A) Multiplicity ➔ Graphic Connection

Even Multiplicity **Odd Multiplicity**





(B) Solving if x ε **C**

- Let's expand our number set from real numbers to complex numbers
- Factor and solve $3 – 2x^2 – x^4 = 0$ if x ε **C**
- Factor and solve $3x^3 – 7x^2 + 8x – 2 = 0$ if x ε **C**
- Factor and solve $2x^3 + 14x - 20 = 9x^2 – 5$ if x ε **C**
- Now write each polynomial as a product of its factors
- Explain the graphic significance of your solutions for x

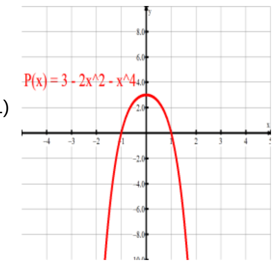
(B) Solving if x ε **C** – Solution to Ex 1

- Factor and solve $3 - 2x^2 - x^4 = 0$ if x ε **C** and then write each polynomial as a product of its factors
- Solutions are $x = \pm 1$ and $x = \pm i\sqrt{3}$
- So rewriting the polynomial in factored form (over the reals) is $P(x) = -(x^2 + 3)(x - 1)(x + 1)$ and over the complex numbers: $P(x) = -(x-1)(x+1)(x-i\sqrt{3})(x+i\sqrt{3})$

(B) Solving if x ε **C** – Graphic Connection

- With $P(x) = 3 - 2x^2 - x^4$, we can now consider a graphic connection, given that
  $P(x) = -(x^2 + 3)(x - 1)(x + 1)$

  or given that
  $P(x) = -(x-1)(x+1)(x-i\sqrt{3})(x+i\sqrt{3})$

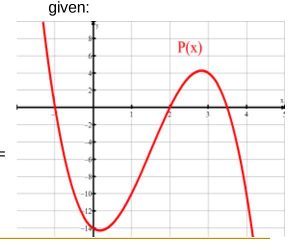


(C) Fundamental Theorem of Algebra

- The fundamental theorem of algebra can be stated in many ways:
  - (a) If P(x) is a polynomial of degree *n* then P(x) will have exactly *n* zeroes (real or complex), some of which may repeat.
  - (b) Every polynomial function of degree $n \geq 1$ has exactly *n* complex zeroes, counting multiplicities
  - (c) If P(x) has a nonreal root, $a+b\mathbf{i}$, where $b \neq 0$, then its conjugate, $a-b\mathbf{i}$ is also a root
  - (d) Every polynomial can be factored (over the real numbers) into a product of linear factors and irreducible quadratic factors
- What does it all mean → we can solve EVERY polynomial (it may be REALLY difficult, but it can be done!)

(D) Using the FTA

- Write an equation of a polynomial whose roots are x = 1, x = 2 and x = ¾
- Write the equation of the polynomial whose roots are 1, -2, -4, & 6 and a point (-1, -84)
- Write the equation of a polynomial whose roots are x = 2 (with a multiplicity of 2) as well as $x = -1 \pm \sqrt{2}$
- Write the equation of a polynomial whose graph is given:



(D) Using the FTA

- Given that 1 – 3i is a root of $x^4 - 4x^3 + 13x^2 - 18x - 10 = 0$, find the remaining roots.
- Write an equation of a third degree polynomial whose given roots are 1 and *i*. Additionally, the polynomial passes through (0,5)
- Write the equation of a quartic wherein you know that one root is 2 – *i* and that the root x = 3 has a multiplicity of 2.

(E) Further Examples

- The equation $x^3 - 3x^2 - 10x + 24 = 0$ has roots of 2, *h*, and *k*. Determine a quadratic equation whose roots are *h* – *k* and *hk*.
- The 5th degree polynomial, f(x), is divisible by x^3 and f(x) – 1 is divisible by $(x - 1)^3$. Find f(x).
- Find the polynomial p(x) with integer coefficients such that one solution of the equation p(x)=0 is $1+\sqrt{2}+\sqrt{3}$.

(K) Factoring Polynomials – The Remainder Theorem - Examples

- Find k so that when $x^2 + 8x + k$ is divided by $x - 2$, the remainder is 3
- Find the value of k so that when $x^3 + 5x^2 + 6x + 11$ is divided by $x + k$, the remainder is 3
- When $P(x) = ax^3 – x^2 - x + b$ is divided by $x - 1$, the remainder is 6. When $P(x)$ is divided by $x + 2$, the remainder is 9. What are the values of a and b?
- Use the remainder theorem to determine if $(x – 4)$ is a factor of $P(x)$ if $P(x) = x^4 – 16x^2 – 2x + 6$

(L) Factoring Polynomials – the Rational Root Theorem

- Our previous examples were slightly misleading … as in too easy (leading coefficient was deliberately 1)
- Consider this example ➔ $-12x^3 + 20x^2 + 33x – 20$ which when factored becomes $(2x-1)(3x+4)(5-2x)$ so the roots would be ½, -4/3, and 5/2
- Make the following observation ➔ that the numerator of the roots (1, -4, 5) are factors of the constant term (-20) while the denominator of the roots (2,3,2) are factors of the leading coefficient (-12)
- We can test this idea with other polynomials ➔ we will find the same pattern ➔ that the roots are in fact some combination of the factors of the leading coefficient and the constant term

(L) Factoring Polynomials – the Rational Root Theorem

- Our previous observation (although limited in development) leads to the following theorem:
- Given $P(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x^1 + a_0$, if $P(x) = 0$ has a rational root of the form a/b and a/b is in lowest terms, then a must be a divisor of a_0 and b must be a divisor of a_n

(L) Factoring Polynomials – the Rational Root Theorem

- So what does this theorem mean?
- If we want to factor $P(x) = 2x^3 – 5x^2 + 22x – 10$, then we first need to find a value a/b such that $P(a/b) = 0$
- So the factors of the leading coefficient are $\{\pm1, \pm2\}$ which are then the possible values for b
- The factors of the constant term, -10, are $\{\pm1, \pm2, \pm5, \pm10\}$ which are then the possible values for a
- Thus the possible ratios a/b which we can test to find the factors are $\{\pm1, \pm½, \pm2, \pm5/2, \pm5, \pm10\}$
- As it then turns out, P(½) turns out to give P(x) = 0, meaning that $(x – ½)$ or $(2x – 1)$ is a factor of P(x)

(M) Factoring Polynomials – the Rational Root Theorem - Examples

- Ex 1. To factor $P(x) = 2x^3 – 9x^2 + 7x + 6$, what values of x could you test according to the RRT
- Now factor P(x)
- Ex 2. To factor $P(x) = 3x^3 – 7x^2 + 8x – 2$ what values of x could you test according to the RRT
- Now factor P(x)
- ex 3 ➔ Graph $f(x) = 3x^3 + x^2 - 22x - 24$ using intercepts, points, and end behaviour. Approximate turning points, max/min points, and intervals of increase and decrease.

Homework

- Textbook, S7.5, p463-464, Q17,19,27,28,31,32,38,43,45,46,48,49,50
- Do some with & some without the TI-84