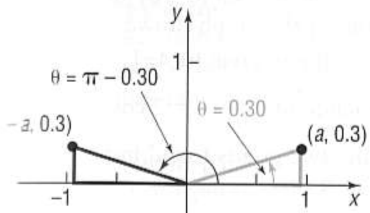
EXAMPLE 6 Solving a Trigonometric Equation with a Calculator

Use a calculator to solve the equation $\sin\theta = 0.3, 0 \le \theta < 2\pi$. Express any solutions in radians, rounded to two decimal places.

Solution To solve $\sin\theta = 0.3$ on a calculator, first set the mode to radians. Then use the $\boxed{\sin^{-1}}$ key to obtain

$$\theta = \sin^{-1}(0.3) \approx 0.3046927$$

Figure 29



Rounded to two decimal places, $\theta = \sin^{-1}(0.3) = 0.30$ radian. Because of the definition of $y = \sin^{-1} x$, the angle θ that we obtain is the angle $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ for which $\sin\theta = 0.3$. Another angle for which $\sin\theta = 0.3$ is $\pi - 0.30$. See Figure 29. The angle $\pi - 0.30$ is the angle in quadrant II, where $\sin\theta = 0.3$. The solutions for $\sin\theta = 0.3, 0 \le \theta < 2\pi$, are

$$\theta = 0.30 \text{ radian} \quad \text{and} \quad \theta = \pi - 0.30 \approx 2.84 \text{ radians}$$

The solution set is $\{0.30, 2.84\}$.

WARNING Example 6 illustrates that caution must be exercised when solving trigonometric equations on a calculator. Remember that the calculator supplies an angle only within the restrictions of the definition of the inverse trigonometric function. To find the remaining solutions, you must identify other quadrants, if any, in which a solution may be located.

Now Work PROBLEM 41

8.7 Assess Your Understanding

Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. Solve: $3x - 5 = -x + 1$ (pp. 86–92)

2. $\sin\left(\frac{\pi}{4}\right) =$ _____; $\cos\left(\frac{8\pi}{3}\right) =$ _____ (pp. 529–532 and 540–548)

Concepts and Vocabulary

3. Two solutions of the equation $\sin\theta = \frac{1}{2}$ are _____ and _____.

4. All the solutions of the equation $\sin\theta = \frac{1}{2}$ are _____.

5. ***True or False*** Most trigonometric equations have unique solutions.

6. ***True or False*** The equation $\sin\theta = 2$ has a real solution that can be found using a calculator.

Skill Building

In Problems 7–30, solve each equation on the interval $0 \le \theta < 2\pi$.

7. $2\sin\theta + 3 = 2$
8. $1 - \cos\theta = \frac{1}{2}$
9. $4\cos^2\theta = 1$
10. $\tan^2\theta = \frac{1}{3}$
11. $2\sin^2\theta - 1 = 0$
12. $4\cos^2\theta - 3 = 0$
13. $\sin(3\theta) = -1$
14. $\tan\frac{\theta}{2} = \sqrt{3}$
15. $\cos(2\theta) = -\frac{1}{2}$
16. $\tan(2\theta) = -1$
17. $\sec\frac{3\theta}{2} = -2$
18. $\cot\frac{2\theta}{3} = -\sqrt{3}$
19. $2\sin\theta + 1 = 0$
20. $\cos\theta + 1 = 0$
21. $\tan\theta + 1 = 0$
22. $\sqrt{3}\cot\theta + 1 = 0$
23. $4\sec\theta + 6 = -2$
24. $5\csc\theta - 3 = 2$
25. $3\sqrt{2}\cos\theta + 2 = -1$
26. $4\sin\theta + 3\sqrt{3} = \sqrt{3}$
27. $\cos\left(2\theta - \frac{\pi}{2}\right) = -1$
28. $\sin\left(3\theta + \frac{\pi}{18}\right) = 1$
29. $\tan\left(\frac{\theta}{2} + \frac{\pi}{3}\right) = 1$
30. $\cos\left(\frac{\theta}{3} - \frac{\pi}{4}\right) = \frac{1}{2}$

In Problems 31–40, solve each equation. Give a general formula for all the solutions. List six solutions.

31. $\sin\theta = \frac{1}{2}$
32. $\tan\theta = 1$
33. $\tan\theta = -\frac{\sqrt{3}}{3}$
34. $\cos\theta = -\frac{\sqrt{3}}{2}$
35. $\cos\theta = 0$

36. $\sin\theta = \frac{\sqrt{2}}{2}$ **37.** $\cos(2\theta) = -\frac{1}{2}$ **38.** $\sin(2\theta) = -1$ **39.** $\sin\frac{\theta}{2} = -\frac{\sqrt{3}}{2}$ **40.** $\tan\frac{\theta}{2} = -1$

In Problems 41–52, use a calculator to solve each equation on the interval $0 \le \theta < 2\pi$. Round answers to two decimal places.

41. $\sin\theta = 0.4$ **42.** $\cos\theta = 0.6$ **43.** $\tan\theta = 5$ **44.** $\cot\theta = 2$

45. $\cos\theta = -0.9$ **46.** $\sin\theta = -0.2$ **47.** $\sec\theta = -4$ **48.** $\csc\theta = -3$

49. $5\tan\theta + 9 = 0$ **50.** $4\cot\theta = -5$ **51.** $3\sin\theta - 2 = 0$ **52.** $4\cos\theta + 3 = 0$

Applications and Extensions

53. What are the x-intercepts of the graph of $f(x) = 4\sin^2 x - 3$ on the interval $[0, 2\pi]$?

54. What are the x-intercepts of the graph of $f(x) = 2\cos(3x) + 1$ on the interval $[0, \pi]$?

55. $f(x) = 3\sin x$.
(a) Find the x-intercepts of the graph of f on the interval $[-2\pi, 4\pi]$.
(b) Graph $f(x) = 3\sin x$ on the interval $[-2\pi, 4\pi]$.
(c) Solve $f(x) = \frac{3}{2}$ on the interval $[-2\pi, 4\pi]$. What points are on the graph of f? Label these points on the graph drawn in part (b).
(d) Use the graph drawn in part (b) along with the results of part (c) to determine the values of x such that $f(x) > \frac{3}{2}$ on the interval $[-2\pi, 4\pi]$.

56. $f(x) = 2\cos x$.
(a) Find the x-intercepts of the graph of f on the interval $[-2\pi, 4\pi]$.
(b) Graph $f(x) = 2\cos x$ on the interval $[-2\pi, 4\pi]$.
(c) Solve $f(x) = -\sqrt{3}$ on the interval $[-2\pi, 4\pi]$. What points are on the graph of f? Label these points on the graph drawn in part (b).
(d) Use the graph drawn in part (b) along with the results of part (c) to determine the values of x such that $f(x) < -\sqrt{3}$ on the interval $[-2\pi, 4\pi]$.

57. $f(x) = 4\tan x$.
(a) Solve $f(x) = -4$.
(b) For what values of x is $f(x) < -4$ on the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$?

58. $f(x) = \cot x$.
(a) Solve $f(x) = -\sqrt{3}$.
(b) For what values of x is $f(x) > -\sqrt{3}$ on the interval $(0, \pi)$?

59. (a) Graph $f(x) = 3\sin(2x) + 2$ and $g(x) = \frac{7}{2}$ on the same Cartesian plane for the interval $[0, \pi]$.
(b) Solve $f(x) = g(x)$ on the interval $[0, \pi]$ and label the points of intersection on the graph drawn in part (b).
(c) Solve $f(x) > g(x)$ on the interval $[0, \pi]$.
(d) Shade the region bounded by $f(x) = 3\sin(2x) + 2$ and $g(x) = \frac{7}{2}$ between the two points found in part (b) on the graph drawn in part (a).

60. (a) Graph $f(x) = 2\cos\frac{x}{2} + 3$ and $g(x) = 4$ on the same Cartesian plane for the interval $[0, 4\pi]$.
(b) Solve $f(x) = g(x)$ on the interval $[0, 4\pi]$ and label the points of intersection on the graph drawn in part (b).
(c) Solve $f(x) < g(x)$ on the interval $[0, 4\pi]$.
(d) Shade the region bounded by $f(x) = 2\cos\frac{x}{2} + 3$ and $g(x) = 4$ between the two points found in part (b) on the graph drawn in part (a).

61. (a) Graph $f(x) = -4\cos x$ and $g(x) = 2\cos x + 3$ on the same Cartesian plane for the interval $[0, 2\pi]$.
(b) Solve $f(x) = g(x)$ on the interval $[0, 2\pi]$ and label the points of intersection on the graph drawn in part (b).
(c) Solve $f(x) > g(x)$ on the interval $[0, 2\pi]$.
(d) Shade the region bounded by $f(x) = -4\cos x$ and $g(x) = 2\cos x + 3$ between the two points found in part (b) on the graph drawn in part (a).

62. (a) Graph $f(x) = 2\sin x$ and $g(x) = -2\sin x + 2$ on the same Cartesian plane for the interval $[0, 2\pi]$.
(b) Solve $f(x) = g(x)$ on the interval $[0, 2\pi]$ and label the points of intersection on the graph drawn in part (b).
(c) Solve $f(x) > g(x)$ on the interval $[0, 2\pi]$.
(d) Shade the region bounded by $f(x) = 2\sin x$ and $g(x) = -2\sin x + 2$ between the two points found in part (b) on the graph drawn in part (a).

63. The Ferris Wheel In 1893, George Ferris engineered the Ferris Wheel. It was 250 feet in diameter. If the wheel makes 1 revolution every 40 seconds, then the function

$$h(t) = 125\sin\left(0.157t - \frac{\pi}{2}\right) + 125$$

represents the height h, in feet, of a seat on the wheel as a function of time t, where t is measured in seconds. The ride begins when $t = 0$.
(a) During the first 40 seconds of the ride, at what time t is an individual on the Ferris Wheel exactly 125 feet above the ground?
(b) During the first 80 seconds of the ride, at what time t is an individual on the Ferris Wheel exactly 250 feet above the ground?
(c) During the first 40 seconds of the ride, over what interval of time t is an individual on the Ferris Wheel more than 125 feet above the ground?

64. Tire Rotation The P215/65R15 Cobra Radial G/T tire has a diameter of exactly 26 inches. Suppose that a car's wheel is making 2 revolutions per second (the car is traveling a little less than 10 miles per hour). Then $h(t) = 13\sin\left(4\pi t - \frac{\pi}{2}\right) + 13$ represents the height h (in inches) of a point on the tire as a function of time t (in seconds). The car starts to move when $t = 0$.

(a) During the first second that the car is moving, at what time t is the point on the tire exactly 13 inches above the ground?

(b) During the first second that the car is moving, at what time t is the point on the tire exactly 6.5 inches above the ground?

(c) During the first second that the car is moving, at what time t is the point on the tire more than 13 inches above the ground?

Source: Cobra Tire

65. Holding Pattern An airplane is asked to stay within a holding pattern near Chicago's O'Hare International Airport. The function $d(x) = 70\sin(0.65x) + 150$ represents the distance d, in miles, that the airplane is from the airport at time x, in minutes.

(a) When the plane enters the holding pattern, $x = 0$, how far is it from O'Hare?

(b) During the first 20 minutes after the plane enters the holding pattern, at what time x is the plane exactly 100 miles from the airport?

(c) During the first 20 minutes after the plane enters the holding pattern, at what time x is the plane more than 100 miles from the airport?

(d) While the plane is in the holding pattern, will it ever be within 70 miles of the airport? Why?

66. Projectile Motion A golfer hits a golf ball with an initial velocity of 100 miles per hour. The range R of the ball as a function of the angle θ to the horizontal is given by $R(\theta) = 672\sin(2\theta)$, where R is measured in feet.

(a) At what angle θ should the ball be hit if the golfer wants the ball to travel 450 feet (150 yards)?

(b) At what angle θ should the ball be hit if the golfer wants the ball to travel 540 feet (180 yards)?

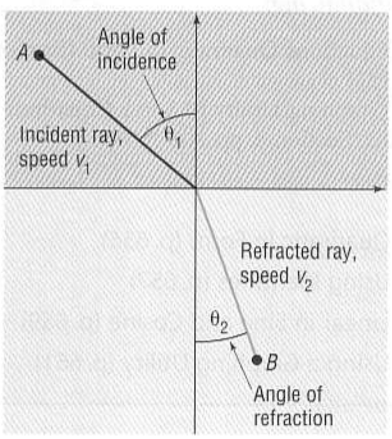
(c) At what angle θ should the ball be hit if the golfer wants the ball to travel at least 480 feet (160 yards)?

(d) Can the golfer hit the ball 720 feet (240 yards)?

The following discussion of Snell's Law of Refraction (named after Willebrord Snell, 1580–1626) is needed for Problems 67–73. Light, sound, and other waves travel at different speeds, depending on the media (air, water, wood, and so on) through which they pass. Suppose that light travels from a point A in one medium, where its speed is v_1, to a point B in another medium, where its speed is v_2. Refer to the figure, where the angle θ_1 is called the **angle of incidence** and the angle θ_2 is the **angle of refraction.** Snell's Law, which can be proved using calculus, states that*

$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$

*The ratio $\frac{v_1}{v_2}$ is called the **index of refraction.** Some values are given in the following table.*



Some Indexes of Refraction

Medium	Index of Refraction†
Water	1.33
Ethyl alcohol (20°C)	1.36
Carbon disulfide	1.63
Air (1 atm and 0°C)	1.00029
Diamond	2.42
Fused quartz	1.46
Glass, crown	1.52
Glass, dense flint	1.66
Sodium chloride	1.54

67. The index of refraction of light in passing from a vacuum into water is 1.33. If the angle of incidence is 40°, determine the angle of refraction.

68. The index of refraction of light in passing from a vacuum into dense flint glass is 1.66. If the angle of incidence is 50°, determine the angle of refraction.

69. Ptolemy, who lived in the city of Alexandria in Egypt during the second century AD, gave the measured values in the table following for the angle of incidence θ_1 and the angle of refraction θ_2 for a light beam passing from air into water. Do these values agree with Snell's Law? If so, what index of refraction results? (These data are interesting as the oldest recorded physical measurements.)†

θ_1	θ_2	θ_1	θ_2
10°	8°	50°	35°0′
20°	15°30′	60°	40°30′
30°	22°30′	70°	45°30′
40°	29°0′	80°	50°0′

* Because this law was also deduced by René Descartes in France, it is also known as Descartes's Law.

† For light of wavelength 589 nanometers, measured with respect to a vacuum. The index with respect to air is negligibly different in most cases.

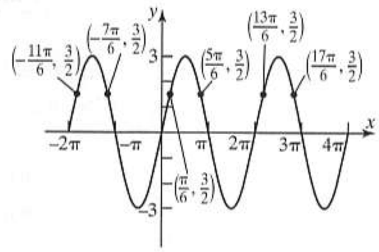
43.
$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$
$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$\sin(\alpha - \beta) + \sin(\alpha + \beta) = 2\sin\alpha\cos\beta$$
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha + \beta) + \sin(\alpha - \beta)]$$

45. $2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2} = 2\cdot\frac{1}{2}\left[\cos\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) + \cos\left(\frac{\alpha+\beta}{2} - \frac{\alpha-\beta}{2}\right)\right] = \cos\frac{2\alpha}{2} + \cos\frac{2\beta}{2} = \cos\alpha + \cos\beta$

8.7 Assess Your Understanding *(page 653)*

3. $\frac{\pi}{6}; \frac{5\pi}{6}$ **4.** $\left\{\theta \middle| \theta = \frac{\pi}{6} + 2\pi k, \theta = \frac{5\pi}{6} + 2\pi k, k \text{ any integer}\right\}$ **5.** F **6.** F **7.** $\left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$ **9.** $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$ **11.** $\left\{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right\}$
13. $\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$ **15.** $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$ **17.** $\left\{\frac{4\pi}{9}, \frac{8\pi}{9}, \frac{16\pi}{9}\right\}$ **19.** $\left\{\frac{7\pi}{6}, \frac{11\pi}{6}\right\}$ **21.** $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ **23.** $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$
25. $\left\{\frac{3\pi}{4}, \frac{5\pi}{4}\right\}$ **27.** $\left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$ **29.** $\left\{\frac{11\pi}{6}\right\}$ **31.** $\left\{\theta \middle| \theta = \frac{\pi}{6} + 2k\pi, \theta = \frac{5\pi}{6} + 2k\pi\right\}; \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{25\pi}{6}, \frac{29\pi}{6}$
33. $\left\{\theta \middle| \theta = \frac{5\pi}{6} + k\pi\right\}; \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}, \frac{29\pi}{6}, \frac{35\pi}{6}$ **35.** $\left\{\theta \middle| \theta = \frac{\pi}{2} + 2k\pi, \theta = \frac{3\pi}{2} + 2k\pi\right\}; \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$
37. $\left\{\theta \middle| \theta = \frac{\pi}{3} + k\pi, \theta = \frac{2\pi}{3} + k\pi\right\}; \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}, \frac{8\pi}{3}$ **39.** $\left\{\theta \middle| \theta = \frac{8\pi}{3} + 4k\pi, \theta = \frac{10\pi}{3} + 4k\pi\right\}; \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{20\pi}{3}, \frac{22\pi}{3}, \frac{32\pi}{3}, \frac{34\pi}{3}$
41. $\{0.41, 2.73\}$ **43.** $\{1.37, 4.51\}$ **45.** $\{2.69, 3.59\}$ **47.** $\{1.82, 4.46\}$ **49.** $\{2.08, 5.22\}$ **51.** $\{0.73, 2.41\}$ **53.** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

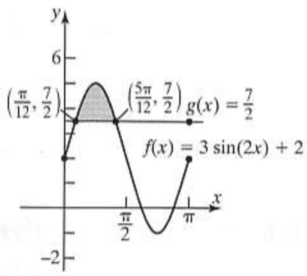
55. (a) $-2\pi, -\pi, 0, \pi, 2\pi, 3\pi, 4\pi$
(b)



(c) $\left\{-\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}\right\}$
(d) $\left\{x \middle| -\frac{11\pi}{6} < x < -\frac{7\pi}{6} \text{ or } \frac{\pi}{6} < x < \frac{5\pi}{6} \text{ or } \frac{13\pi}{6} < x < \frac{17\pi}{6}\right\}$

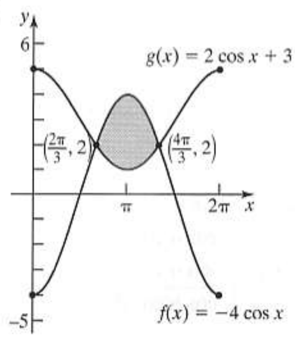
57. (a) $\left\{x \middle| x = -\frac{\pi}{4} + k\pi\right\}$ **(b)** $-\frac{\pi}{2} < x < -\frac{\pi}{4}$ or $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$

59. (a), (d)



(b) $\left\{\frac{\pi}{12}, \frac{5\pi}{12}\right\}$
(c) $\left\{x \middle| \frac{\pi}{12} < x < \frac{5\pi}{12}\right\}$ or $\left(\frac{\pi}{12}, \frac{5\pi}{12}\right)$

61. (a), (d)



(b) $\left\{\frac{2\pi}{3}, \frac{4\pi}{3}\right\}$
(c) $\left\{x \middle| \frac{2\pi}{3} < x < \frac{4\pi}{3}\right\}$ or $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

63. (a) 10 sec; 30 sec **(b)** 20 sec; 60 sec
(c) $10 < x < 30$ or $(10, 30)$
65. (a) 150 mi **(b)** 6.06, 8.44, 15.72, 18.11 min
(c) Before 6.06 min, between 8.44 and 15.72 min, and after 18.11 min **(d)** No
67. 28.90° **69.** Yes; it varies from 1.25 to 1.34.
71. 1.47
73. If θ is the original angle of incidence and ϕ is the angle of refraction, then $\frac{\sin\theta}{\sin\phi} = n_2$. The angle of incidence of the emerging beam is also ϕ, and the index of refraction is $\frac{1}{n_2}$. Thus, θ is the angle of refraction of the emerging beam.

8.8 Assess Your Understanding *(page 661)*

5. $\left\{\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}\right\}$ **7.** $\left\{\frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}\right\}$ **9.** $\left\{0, \frac{\pi}{4}, \frac{5\pi}{4}\right\}$ **11.** $\left\{\frac{\pi}{2}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{3\pi}{2}\right\}$ **13.** $\{\pi\}$ **15.** $\left\{\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}\right\}$ **17.** $\left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$ **19.** $\left\{0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$
21. $\left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ **23.** $\left\{0, \frac{2\pi}{3}, \frac{4\pi}{3}\right\}$ **25.** $\left\{0, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}\right\}$ **27.** $\left\{0, \frac{\pi}{5}, \frac{2\pi}{5}, \frac{3\pi}{5}, \frac{4\pi}{5}, \pi, \frac{6\pi}{5}, \frac{7\pi}{5}, \frac{8\pi}{5}, \frac{9\pi}{5}\right\}$ **29.** $\left\{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}\right\}$ **31.** $\left\{\frac{\pi}{2}\right\}$
33. $\{0\}$ **35.** $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$ **37.** No real solution **39.** No real solution **41.** $\left\{\frac{\pi}{2}, \frac{7\pi}{6}\right\}$ **43.** $\left\{0, \frac{\pi}{3}, \pi, \frac{5\pi}{3}\right\}$ **45.** $\left\{\frac{\pi}{4}\right\}$ **47.** $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$