Since $-\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$, we know that $\cos\alpha \ge 0$. As a result,

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-u^2}$$

Similarly, since $0 \le \beta \le \pi$, we know that $\sin\beta \ge 0$. Then

$$\sin\beta = \sqrt{1-\cos^2\beta} = \sqrt{1-v^2}$$

As a result,

$$\begin{aligned}\sin(\sin^{-1}u + \cos^{-1}v) &= \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= uv + \sqrt{1-u^2}\sqrt{1-v^2}\end{aligned}$$

Now Work PROBLEM 83

SUMMARY Sum and Difference Formulas

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \tan(\alpha-\beta) = \frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

8.4 Assess Your Understanding

'Are You Prepared?' *Answers are given at the end of these exercises. If you get a wrong answer, read the pages listed in red.*

1. The distance d from the point $(2, -3)$ to the point $(5, 1)$ is ______. (p. 157)
2. If $\sin\theta = \frac{4}{5}$ and θ is in quadrant II, then $\cos\theta =$ ______. (pp. 546–548)
3. (a) $\sin\frac{\pi}{4}\cdot\cos\frac{\pi}{3} =$ ______. (pp. 529–532)
   (b) $\tan\frac{\pi}{4} - \sin\frac{\pi}{6} =$ ______. (pp. 529–532)

Concepts and Vocabulary

4. $\cos(\alpha+\beta) = \cos\alpha\cos\beta$ ____ $\sin\alpha\sin\beta$.
5. $\sin(\alpha-\beta) = \sin\alpha\cos\beta$ ____ $\cos\alpha\sin\beta$.
6. ***True or False*** $\sin(\alpha+\beta) = \sin\alpha + \sin\beta + 2\sin\alpha\sin\beta$
7. ***True or False*** $\tan 75° = \tan 30° + \tan 45°$
8. ***True or False*** $\cos\left(\frac{\pi}{2} - \theta\right) = \cos\theta$

Skill Building

In Problems 9–20, find the exact value of each expression.

9. $\sin\frac{5\pi}{12}$
10. $\sin\frac{\pi}{12}$
11. $\cos\frac{7\pi}{12}$
12. $\tan\frac{7\pi}{12}$
13. $\cos 165°$
14. $\sin 105°$
15. $\tan 15°$
16. $\tan 195°$
17. $\sin\frac{17\pi}{12}$
18. $\tan\frac{19\pi}{12}$
19. $\sec\left(-\frac{\pi}{12}\right)$
20. $\cot\left(-\frac{5\pi}{12}\right)$

In Problems 21–30, find the exact value of each expression.

21. $\sin 20°\cos 10° + \cos 20°\sin 10°$
22. $\sin 20°\cos 80° - \cos 20°\sin 80°$
23. $\cos 70°\cos 20° - \sin 70°\sin 20°$
24. $\cos 40°\cos 10° + \sin 40°\sin 10°$
25. $\dfrac{\tan 20° + \tan 25°}{1 - \tan 20°\tan 25°}$
26. $\dfrac{\tan 40° - \tan 10°}{1 + \tan 40°\tan 10°}$
27. $\sin\frac{\pi}{12}\cos\frac{7\pi}{12} - \cos\frac{\pi}{12}\sin\frac{7\pi}{12}$
28. $\cos\frac{5\pi}{12}\cos\frac{7\pi}{12} - \sin\frac{5\pi}{12}\sin\frac{7\pi}{12}$
29. $\cos\frac{\pi}{12}\cos\frac{5\pi}{12} + \sin\frac{5\pi}{12}\sin\frac{\pi}{12}$
30. $\sin\frac{\pi}{18}\cos\frac{5\pi}{18} + \cos\frac{\pi}{18}\sin\frac{5\pi}{18}$

In Problems 31–36, find the exact value of each of the following under the given conditions:

(a) $\sin(\alpha + \beta)$ (b) $\cos(\alpha + \beta)$ (c) $\sin(\alpha - \beta)$ (d) $\tan(\alpha - \beta)$

31. $\sin \alpha = \frac{3}{5}, 0 < \alpha < \frac{\pi}{2}$; $\cos \beta = \frac{2\sqrt{5}}{5}, -\frac{\pi}{2} < \beta < 0$

32. $\cos \alpha = \frac{\sqrt{5}}{5}, 0 < \alpha < \frac{\pi}{2}$; $\sin \beta = -\frac{4}{5}, -\frac{\pi}{2} < \beta < 0$

33. $\tan \alpha = -\frac{4}{3}, \frac{\pi}{2} < \alpha < \pi$; $\cos \beta = \frac{1}{2}, 0 < \beta < \frac{\pi}{2}$

34. $\tan \alpha = \frac{5}{12}, \pi < \alpha < \frac{3\pi}{2}$; $\sin \beta = -\frac{1}{2}, \pi < \beta < \frac{3\pi}{2}$

35. $\sin \alpha = \frac{5}{13}, -\frac{3\pi}{2} < \alpha < -\pi$; $\tan \beta = -\sqrt{3}, \frac{\pi}{2} < \beta < \pi$

36. $\cos \alpha = \frac{1}{2}, -\frac{\pi}{2} < \alpha < 0$; $\sin \beta = \frac{1}{3}, 0 < \beta < \frac{\pi}{2}$

37. If $\sin \theta = \frac{1}{3}$, θ in quadrant II, find the exact value of:

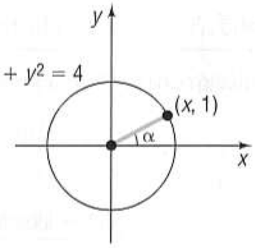
(a) $\cos \theta$ (b) $\sin\left(\theta + \frac{\pi}{6}\right)$ (c) $\cos\left(\theta - \frac{\pi}{3}\right)$ (d) $\tan\left(\theta + \frac{\pi}{4}\right)$

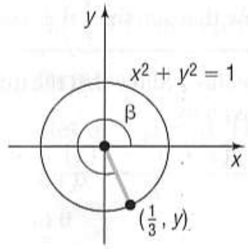
38. If $\cos \theta = \frac{1}{4}$, θ in quadrant IV, find the exact value of:

(a) $\sin \theta$ (b) $\sin\left(\theta - \frac{\pi}{6}\right)$ (c) $\cos\left(\theta + \frac{\pi}{3}\right)$ (d) $\tan\left(\theta - \frac{\pi}{4}\right)$

In Problems 39–44, use the figures to evaluate each function if $f(x) = \sin x$, $g(x) = \cos x$, and $h(x) = \tan x$.

39. $f(\alpha + \beta)$
40. $g(\alpha + \beta)$
41. $g(\alpha - \beta)$
42. $f(\alpha - \beta)$
43. $h(\alpha + \beta)$
44. $h(\alpha - \beta)$





In Problems 45–70, establish each identity.

45. $\sin\left(\frac{\pi}{2} + \theta\right) = \cos \theta$

46. $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$

47. $\sin(\pi - \theta) = \sin \theta$

48. $\cos(\pi - \theta) = -\cos \theta$

49. $\sin(\pi + \theta) = -\sin \theta$

50. $\cos(\pi + \theta) = -\cos \theta$

51. $\tan(\pi - \theta) = -\tan \theta$

52. $\tan(2\pi - \theta) = -\tan \theta$

53. $\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos \theta$

54. $\cos\left(\frac{3\pi}{2} + \theta\right) = \sin \theta$

55. $\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta$

56. $\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2 \cos \alpha \cos \beta$

57. $\frac{\sin(\alpha + \beta)}{\sin \alpha \cos \beta} = 1 + \cot \alpha \tan \beta$

58. $\frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta} = \tan \alpha + \tan \beta$

59. $\frac{\cos(\alpha + \beta)}{\cos \alpha \cos \beta} = 1 - \tan \alpha \tan \beta$

60. $\frac{\cos(\alpha - \beta)}{\sin \alpha \cos \beta} = \cot \alpha + \tan \beta$

61. $\frac{\sin(\alpha + \beta)}{\sin(\alpha - \beta)} = \frac{\tan \alpha + \tan \beta}{\tan \alpha - \tan \beta}$

62. $\frac{\cos(\alpha + \beta)}{\cos(\alpha - \beta)} = \frac{1 - \tan \alpha \tan \beta}{1 + \tan \alpha \tan \beta}$

63. $\cot(\alpha + \beta) = \frac{\cot \alpha \cot \beta - 1}{\cot \beta + \cot \alpha}$

64. $\cot(\alpha - \beta) = \frac{\cot \alpha \cot \beta + 1}{\cot \beta - \cot \alpha}$

65. $\sec(\alpha + \beta) = \frac{\csc \alpha \csc \beta}{\cot \alpha \cot \beta - 1}$

66. $\sec(\alpha - \beta) = \frac{\sec \alpha \sec \beta}{1 + \tan \alpha \tan \beta}$

67. $\sin(\alpha - \beta) \sin(\alpha + \beta) = \sin^2 \alpha - \sin^2 \beta$

68. $\cos(\alpha - \beta) \cos(\alpha + \beta) = \cos^2 \alpha - \sin^2 \beta$

69. $\sin(\theta + k\pi) = (-1)^k \sin \theta$, k any integer

70. $\cos(\theta + k\pi) = (-1)^k \cos \theta$, k any integer

In Problems 71–82, find the exact value of each expression.

71. $\sin\left(\sin^{-1} \frac{1}{2} + \cos^{-1} 0\right)$

72. $\sin\left(\sin^{-1} \frac{\sqrt{3}}{2} + \cos^{-1} 1\right)$

73. $\sin\left[\sin^{-1} \frac{3}{5} - \cos^{-1}\left(-\frac{4}{5}\right)\right]$

74. $\sin\left[\sin^{-1}\left(-\frac{4}{5}\right) - \tan^{-1}\frac{3}{4}\right]$

75. $\cos\left(\tan^{-1}\frac{4}{3} + \cos^{-1}\frac{5}{13}\right)$

76. $\cos\left[\tan^{-1}\frac{5}{12} - \sin^{-1}\left(-\frac{3}{5}\right)\right]$

77. $\cos\left(\sin^{-1}\frac{5}{13} - \tan^{-1}\frac{3}{4}\right)$

78. $\cos\left(\tan^{-1}\frac{4}{3} + \cos^{-1}\frac{12}{13}\right)$

79. $\tan\left(\sin^{-1}\frac{3}{5} + \frac{\pi}{6}\right)$

80. $\tan\left(\frac{\pi}{4} - \cos^{-1}\frac{3}{5}\right)$

81. $\tan\left(\sin^{-1}\frac{4}{5} + \cos^{-1}1\right)$

82. $\tan\left(\cos^{-1}\frac{4}{5} + \sin^{-1}1\right)$

In Problems 83–88, write each trigonometric expression as an algebraic expression containing u and v. Give the restrictions required on u and v.

83. $\cos(\cos^{-1} u + \sin^{-1} v)$

84. $\sin(\sin^{-1} u - \cos^{-1} v)$

85. $\sin(\tan^{-1} u - \sin^{-1} v)$

86. $\cos(\tan^{-1} u + \tan^{-1} v)$

87. $\tan(\sin^{-1} u - \cos^{-1} v)$

88. $\sec(\tan^{-1} u + \cos^{-1} v)$

Applications and Extensions

89. Show that $\sin^{-1} v + \cos^{-1} v = \frac{\pi}{2}$.

90. Show that $\tan^{-1} v + \cot^{-1} v = \frac{\pi}{2}$.

91. Show that $\tan^{-1}\left(\frac{1}{v}\right) = \frac{\pi}{2} - \tan^{-1} v$, if $v > 0$.

92. Show that $\cot^{-1} e^{v} = \tan^{-1} e^{-v}$.

93. Show that $\sin(\sin^{-1} v + \cos^{-1} v) = 1$.

94. Show that $\cos(\sin^{-1} v + \cos^{-1} v) = 0$.

95. Calculus Show that the difference quotient for $f(x) = \sin x$ is given by

$$\frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h} = \cos x \cdot \frac{\sin h}{h} - \sin x \cdot \frac{1 - \cos h}{h}$$

96. Calculus Show that the difference quotient for $f(x) = \cos x$ is given by

$$\frac{f(x+h) - f(x)}{h} = \frac{\cos(x+h) - \cos x}{h} = -\sin x \cdot \frac{\sin h}{h} - \cos x \cdot \frac{1 - \cos h}{h}$$

97. One, Two, Three

(a) Show that $\tan(\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3) = 0$.

(b) Conclude from part (a) that

$$\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi.$$

Source: College Mathematics Journal, Vol. 37, No. 3, May 2006

98. Electric Power In an alternating current (ac) circuit, the instantaneous power p at time t is given by

$$p(t) = V_m I_m \cos\phi \sin^2(\omega t) - V_m I_m \sin\phi \sin(\omega t)\cos(\omega t)$$

Show that this is equivalent to

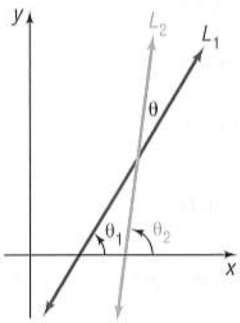
$$p(t) = V_m I_m \sin(\omega t)\sin(\omega t - \phi)$$

Source: HyperPhysics, hosted by Georgia State University

99. Geometry: Angle Between Two Lines Let L_1 and L_2 denote two nonvertical intersecting lines, and let θ denote the acute angle between L_1 and L_2 (see the figure). Show that

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of L_1 and L_2, respectively. [**Hint:** Use the facts that $\tan\theta_1 = m_1$ and $\tan\theta_2 = m_2$.]



100. If $\alpha + \beta + \gamma = 180°$ and

$$\cot\theta = \cot\alpha + \cot\beta + \cot\gamma, \quad 0 < \theta < 90°$$

show that

$$\sin^3\theta = \sin(\alpha - \theta)\sin(\beta - \theta)\sin(\gamma - \theta)$$

101. If $\tan\alpha = x + 1$ and $\tan\beta = x - 1$, show that

$$2\cot(\alpha - \beta) = x^2$$

Discussion and Writing

102. Discuss the following derivation:

$$\tan\left(\theta + \frac{\pi}{2}\right) = \frac{\tan\theta + \tan\frac{\pi}{2}}{1 - \tan\theta\tan\frac{\pi}{2}} = \frac{\frac{\tan\theta}{\tan\frac{\pi}{2}} + 1}{\frac{1}{\tan\frac{\pi}{2}} - \tan\theta} = \frac{0 + 1}{0 - \tan\theta} = \frac{1}{-\tan\theta} = -\cot\theta$$

Can you justify each step?

13. $\sec\theta = 3 \quad \sin\theta > 0$

14. $\csc\theta = -\sqrt{5} \quad \cos\theta < 0$

15. $\cot\theta = -2 \quad \sec\theta < 0$

16. $\sec\theta = 2 \quad \csc\theta < 0$

17. $\tan\theta = -3 \quad \sin\theta < 0$

18. $\cot\theta = 3 \quad \cos\theta < 0$

In Problems 19–28, use the Half-angle Formulas to find the exact value of each expression.

19. $\sin 22.5°$

20. $\cos 22.5°$

21. $\tan\dfrac{7\pi}{8}$

22. $\tan\dfrac{9\pi}{8}$

23. $\cos 165°$

24. $\sin 195°$

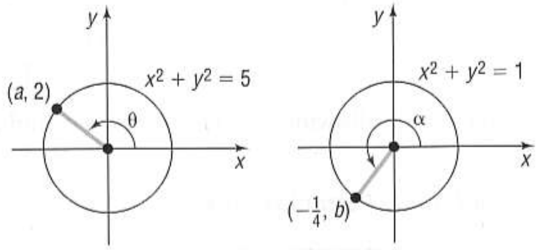
25. $\sec\dfrac{15\pi}{8}$

26. $\csc\dfrac{7\pi}{8}$

27. $\sin\left(-\dfrac{\pi}{8}\right)$

28. $\cos\left(-\dfrac{3\pi}{8}\right)$

In Problems 29–40, use the figures to evaluate each function given that $f(x) = \sin x$, $g(x) = \cos x$, and $h(x) = \tan x$.



29. $f(2\theta)$

30. $g(2\theta)$

31. $g\left(\dfrac{\theta}{2}\right)$

32. $f\left(\dfrac{\theta}{2}\right)$

33. $h(2\theta)$

34. $h\left(\dfrac{\theta}{2}\right)$

35. $g(2\alpha)$

36. $f(2\alpha)$

37. $f\left(\dfrac{\alpha}{2}\right)$

38. $g\left(\dfrac{\alpha}{2}\right)$

39. $h\left(\dfrac{\alpha}{2}\right)$

40. $h(2\alpha)$

41. Show that $\sin^4\theta = \dfrac{3}{8} - \dfrac{1}{2}\cos(2\theta) + \dfrac{1}{8}\cos(4\theta)$.

42. Show that $\sin(4\theta) = (\cos\theta)(4\sin\theta - 8\sin^3\theta)$.

43. Develop a formula for $\cos(3\theta)$ as a third-degree polynomial in the variable $\cos\theta$.

44. Develop a formula for $\cos(4\theta)$ as a fourth-degree polynomial in the variable $\cos\theta$.

45. Find an expression for $\sin(5\theta)$ as a fifth-degree polynomial in the variable $\sin\theta$.

46. Find an expression for $\cos(5\theta)$ as a fifth-degree polynomial in the variable $\cos\theta$.

In Problems 47–68, establish each identity.

47. $\cos^4\theta - \sin^4\theta = \cos(2\theta)$

48. $\dfrac{\cot\theta - \tan\theta}{\cot\theta + \tan\theta} = \cos(2\theta)$

49. $\cot(2\theta) = \dfrac{\cot^2\theta - 1}{2\cot\theta}$

50. $\cot(2\theta) = \dfrac{1}{2}(\cot\theta - \tan\theta)$

51. $\sec(2\theta) = \dfrac{\sec^2\theta}{2 - \sec^2\theta}$

52. $\csc(2\theta) = \dfrac{1}{2}\sec\theta\csc\theta$

53. $\cos^2(2u) - \sin^2(2u) = \cos(4u)$

54. $(4\sin u\cos u)(1 - 2\sin^2 u) = \sin(4u)$

55. $\dfrac{\cos(2\theta)}{1 + \sin(2\theta)} = \dfrac{\cot\theta - 1}{\cot\theta + 1}$

56. $\sin^2\theta\cos^2\theta = \dfrac{1}{8}[1 - \cos(4\theta)]$

57. $\sec^2\dfrac{\theta}{2} = \dfrac{2}{1 + \cos\theta}$

58. $\csc^2\dfrac{\theta}{2} = \dfrac{2}{1 - \cos\theta}$

59. $\cot^2\dfrac{v}{2} = \dfrac{\sec v + 1}{\sec v - 1}$

60. $\tan\dfrac{v}{2} = \csc v - \cot v$

61. $\cos\theta = \dfrac{1 - \tan^2\dfrac{\theta}{2}}{1 + \tan^2\dfrac{\theta}{2}}$

62. $1 - \dfrac{1}{2}\sin(2\theta) = \dfrac{\sin^3\theta + \cos^3\theta}{\sin\theta + \cos\theta}$

63. $\dfrac{\sin(3\theta)}{\sin\theta} - \dfrac{\cos(3\theta)}{\cos\theta} = 2$

64. $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} - \dfrac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} = 2\tan(2\theta)$

65. $\tan(3\theta) = \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$

66. $\tan\theta + \tan(\theta + 120°) + \tan(\theta + 240°) = 3\tan(3\theta)$

67. $\ln|\sin\theta| = \dfrac{1}{2}(\ln|1 - \cos(2\theta)| - \ln 2)$

68. $\ln|\cos\theta| = \dfrac{1}{2}(\ln|1 + \cos(2\theta)| - \ln 2)$

In Problems 69–80, find the exact value of each expression.

69. $\sin\left(2\sin^{-1}\frac{1}{2}\right)$ **70.** $\sin\left[2\sin^{-1}\frac{\sqrt{3}}{2}\right]$ **71.** $\cos\left(2\sin^{-1}\frac{3}{5}\right)$ **72.** $\cos\left(2\cos^{-1}\frac{4}{5}\right)$

73. $\tan\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]$ **74.** $\tan\left(2\tan^{-1}\frac{3}{4}\right)$ **75.** $\sin\left(2\cos^{-1}\frac{4}{5}\right)$ **76.** $\cos\left[2\tan^{-1}\left(-\frac{4}{3}\right)\right]$

77. $\sin^2\left(\frac{1}{2}\cos^{-1}\frac{3}{5}\right)$ **78.** $\cos^2\left(\frac{1}{2}\sin^{-1}\frac{3}{5}\right)$ **79.** $\sec\left(2\tan^{-1}\frac{3}{4}\right)$ **80.** $\csc\left[2\sin^{-1}\left(-\frac{3}{5}\right)\right]$

Applications and Extensions

81. Laser Projection In a laser projection system, the **optical** or **scanning angle** θ is related to the throw distance D from the scanner to the screen and the projected image width W by the equation $D = \dfrac{\frac{1}{2}W}{\csc\theta - \cot\theta}$.

(a) Show that the projected image width is given by $W = 2D\tan\frac{\theta}{2}$.

(b) Find the optical angle if the throw distance is 15 feet and the projected image width is 6.5 feet.

Source: *Pangolin Laser Systems, Inc.*

82. Product of Inertia The **product of inertia** for an area about inclined axes is given by the formula

$$I_{uv} = I_x\sin\theta\cos\theta - I_y\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta).$$

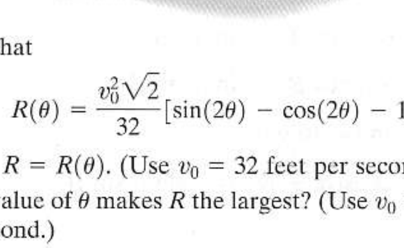
Show that this is equivalent to

$$I_{uv} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta.$$

Source: Adapted from Hibbeler, *Engineering Mechanics: Statics*, 10th ed., Prentice Hall © 2004.

83. Projectile Motion An object is propelled upward at an angle θ, $45° < \theta < 90°$, to the horizontal with an initial velocity of v_0 feet per second from the base of a plane that makes an angle of 45°; with the horizontal. See the illustration. If air resistance is ignored, the distance R that it travels up the inclined plane is given by the function

$$R(\theta) = \frac{v_0^2\sqrt{2}}{16}\cos\theta(\sin\theta - \cos\theta)$$



(a) Show that

$$R(\theta) = \frac{v_0^2\sqrt{2}}{32}[\sin(2\theta) - \cos(2\theta) - 1]$$

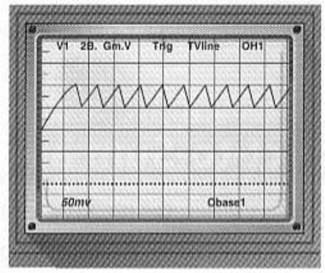
(b) Graph $R = R(\theta)$. (Use $v_0 = 32$ feet per second.)

(c) What value of θ makes R the largest? (Use $v_0 = 32$ feet per second.)

84. Sawtooth Curve An oscilloscope often displays a sawtooth curve. This curve can be approximated by sinusoidal curves of varying periods and amplitudes. A first approximation to the sawtooth curve is given by

$$y = \frac{1}{2}\sin(2\pi x) + \frac{1}{4}\sin(4\pi x)$$

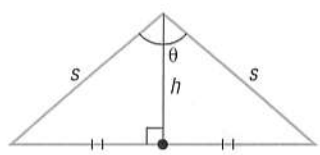
Show that $y = \sin(2\pi x)\cos^2(\pi x)$.



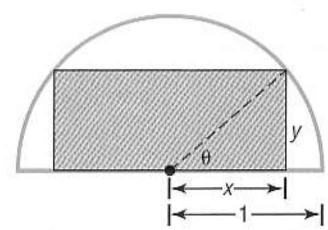
85. Area of an Isosceles Triangle Show that the area A of an isosceles triangle whose equal sides are of length s and θ is the angle between them is

$$\frac{1}{2}s^2\sin\theta$$

[**Hint:** See the illustration. The height h bisects the angle θ and is the perpendicular bisector of the base.]



86. Geometry A rectangle is inscribed in a semicircle of radius 1. See the illustration.



(a) Express the area A of the rectangle as a function of the angle θ shown in the illustration.

(b) Show that $A(\theta) = \sin(2\theta)$.

(c) Find the angle θ that results in the largest area A.

(d) Find the dimensions of this largest rectangle.

87. If $x = 2\tan\theta$, express $\sin(2\theta)$ as a function of x.

88. If $x = 2\tan\theta$, express $\cos(2\theta)$ as a function of x.