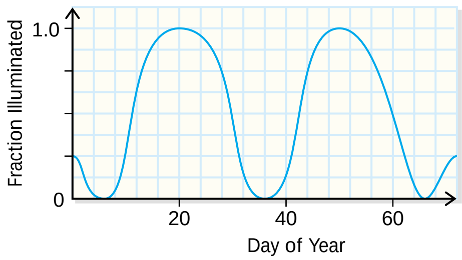
5.7 Modelling Periodic Phenomena

In section 5.1, you worked with this table that gives the fraction of the moon that is visible at midnight as the new millennium began. You drew a scatter plot and the curve of best fit to model the waxing and waning of the moon.

Day of the Year	1	2	3	4	5	6	10	15	20	21
Fraction of Moon Visible	0.25	0.18	0.11	0.06	0.02	0.00	0.11	0.57	0.99	1.00

Day of the Year	25	30	35	40	45	50	55	60	65	66
Fraction of Moon Visible	0.80	0.32	0.02	0.14	0.64	1.00	0.77	0.31	0.01	0.00

Source: US Naval Observatory, Washington.



You found that this data represents a periodic phenomenon with the following properties:

- The period is about 29.5 days.
- The "full" moon is fully visible when the maximum value is 1.0. The "new" moon is not visible when the minimum value is 0.
- The axis of the curve is the horizontal line $y = 0.5$.
- The amplitude of the curve is $\frac{\text{maximum} - \text{minimum}}{2} = 0.5$.

You know that a sinusoidal model of this data is $y = a \sin k(\theta + b) + d$. Find the values of a, k, b, and d to complete the model.

The amplitude, a, is 0.5.

The period of $y = \sin\theta$ is 2π and the period of this function is 29.5 days. Then

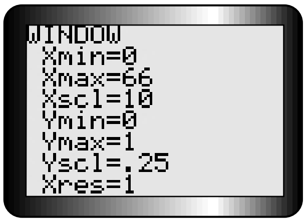
$$k\theta = 2\pi$$
$$29.5k = 2\pi$$
$$k = \frac{2\pi}{29.5}$$

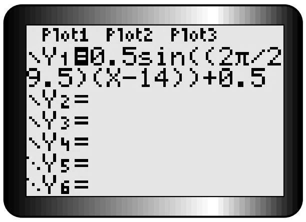
Compare the graph here to $y = \sin \theta$. The phase shift, b, is about 14 days to the left. Therefore, $b = -14$.

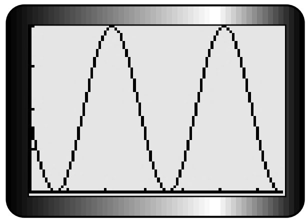
The equation for the axis of the curve is $y = 0.5$. Then the vertical translation, d, is 0.5.

Substituting in $y = a \sin k(\theta + b) + d$ gives $y = 0.5 \sin \frac{2\pi}{29.5}(t - 14) + 0.5$, where t represents the day of the year.

Verify this model by graphing the equation with a graphing calculator. Before graphing, set the mode to radian.







Example 1

Write the model of the waxing and waning of the moon at the turn of the millennium as a cosine function.

Solution

The cosine function has the form $y = a \cos k(t + b) + d$, and $a = 0.5$, $k = \frac{2\pi}{29.5}$, and $d = 0.5$. The variable t represents the day of the year.

To determine the value of b, examine the original graph on page 460 to identify where the cosine curve could start. The cosine pattern begins at the maximum value. In this case, $t = 21$ days. Then $b = -21$.

Therefore, the model is $y = 0.5 \cos \frac{2\pi}{29.5}(t - 21) + 0.5$.

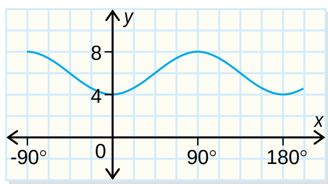
Example 2

Determine the function that is the simplest model of the following data.

Independent Variable	−90°	−45°	0°	45°	90°	125°	180°
Dependent Variable	8	6	4	6	8	6	4

Solution

First decide what type of model could represent the data. Draw a sketch to help you decide.



The curve in the graph appears to represent a sine function or a cosine function.

Case 1: The Sine Model

For the sine model, $y = a \sin k(\theta + b) + d$.

The maximum value is 8 and the minimum value is 4.

The axis of the graph is midway between the maximum and minimum values.

The equation of the axis $y = \frac{8 + 4}{2}$ or $y = 6$.

Then d, the vertical translation, is 6.

The amplitude, a, is half the difference between the maximum and minimum values, $\frac{8 - 4}{2} = 2$. Then $a = 2$.

The period is 180°. There are two complete cycles for each complete cycle of $y = \sin \theta$. The horizontal compression is a factor of $\frac{1}{2}$, so $k = 2$.

Another way to determine k is to solve $k\theta = 360°$ when $\theta = 180°$.

$$\begin{aligned} k\theta &= 360° \\ k(180°) &= 360° \\ k &= \frac{360°}{180°} \\ k &= 2 \end{aligned}$$

The part of this graph that corresponds to the sine function starts at 45°, which is a shift to the right. Then b is $-45°$.

Now substitute $a = 2$, $k = 2$, $b = -45°$, and $d = 6$ in $y = a \sin k(\theta + b) + d$ to get $y = 2 \sin 2(\theta - 45°) + 6$.

Case 2: The Cosine Model

For the cosine model, $y = a \cos k(\theta + b) + d$.

The horizontal compression, $k = 2$, and the vertical translation, $d = 6$, do not change.

The cosine curve reflects about the line $y = 6$. For $a = -2$, the phase angle, b, is 0.

Then the cosine model is $y = -2 \cos 2\theta + 6$.

The simplest model for this data is the one in Case 2, $y = -2 \cos 2\theta + 6$.

For both cases, the maximum and minimum values appeared in the original data. Sometimes these are not given in the data and must be projected from the graph.

Consolidate Your Understanding

1. How do you decide that a trigonometric model best represents the data?

2. How do you decide which trigonometric model to use?

3. How can you verify that the model represents the data?

Focus 5.7

Key Ideas

- You can obtain the trigonometric models $y = a \sin k(\theta + b) + d$ and $y = a \cos k(\theta + b) + d$ by first graphing periodic sinusoidal data.
- You can model and solve many real-life problems using $y = a \sin k(\theta + b) + d$ or $y = a \cos k(\theta + b) + d$.
- The values of a, k, b, and d are found by determining the transformations that must be applied to $y = \sin \theta$ or $y = \cos \theta$, respectively, to obtain the graph of the data.

Practise, Apply, Solve 5.7

A

1. Graph the data in each table. Then determine the model $y = a \sin \theta$ of the data.

(a)

θ	−180°	−90°	0°	90°	180°	270°	360°
y	0	−3	0	3	0	−3	0

(b)

θ	$-\pi$	$-\frac{\pi}{2}$	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
y	0	2	0	−2	0	2	0

2. Graph the data. Then determine the model $y = \cos k\theta$ of the data.

(a)

θ	−90°	−45°	0°	45°	90°	135°	180°
y	−1	0	1	0	−1	0	1

(b)

θ	-2π	$-\pi$	0	π	2π	3π	4π
y	−1	0	1	0	−1	0	1

3. Graph the data. Then determine the model $y = \sin(\theta + b)$ of the data.

(a)

θ	30°	120°	210°	300°	390°
y	0	1	0	−1	0

(b)

θ	−45°	45°	135°	225°	315°
y	0	1	0	−1	0

4. Graph the data. Then determine the model $y = \cos(\theta + b)$ of the data.

(a)

θ	$\frac{2\pi}{6}$	$\frac{5\pi}{6}$	$\frac{8\pi}{6}$	$\frac{11\pi}{6}$	$\frac{14\pi}{6}$
y	1	0	−1	0	1

(b)

θ	$-\frac{\pi}{6}$	$\frac{2\pi}{6}$	$\frac{5\pi}{6}$	$\frac{8\pi}{6}$	$\frac{11\pi}{6}$
y	1	0	−1	0	1

5. Use the ordered pairs $(-30°, 0)$, $(15°, 1)$, $(60°, 0)$, $(105°, -1)$, and $(150°, 0)$ to determine an equation in the form $y = \sin k(\theta + b)$.

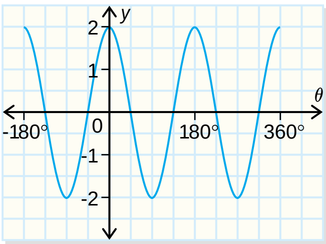
6. The ordered pairs $(0°, 5)$, $(90°, 6)$, $(180°, 5)$, $(270°, 4)$, and $(360°, 5)$ satisfy the equation $y = \sin \theta + d$. Determine d.

7. The ordered pairs $(0, -8)$, $\left(\frac{\pi}{2}, -9\right)$, $(\pi, -10)$, $\left(\frac{3\pi}{2}, -9\right)$, and $(2\pi, -8)$ satisfy the equation $y = \cos \theta + d$. Determine d.

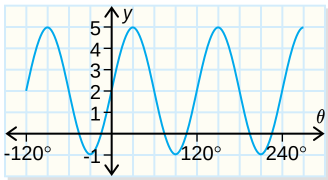
B

8. **Knowledge and Understanding:** Find the sine function $y = a \sin k(\theta + b) + d$ for each graph.

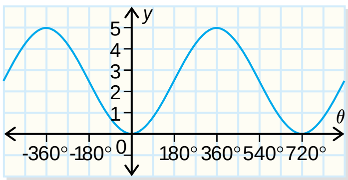
(a)



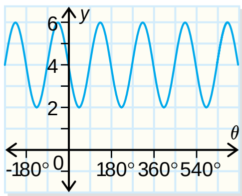
(b)



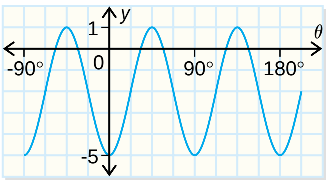
(c)



(d)



(e)



9. Find the cosine function $y = a \cos k(\theta + b) + d$ for each graph in question 8.

10. Determine the sine function that models the given information.

	Amplitude	Period	Phase Shift	Vertical Shift
(a)	3	2π	$\frac{\pi}{4}$	-1
(b)	$\frac{1}{2}$	π	$-\frac{\pi}{3}$	2
(c)	2	$\frac{\pi}{2}$	$\frac{\pi}{6}$	3
(d)	-2	4π	$\frac{\pi}{8}$	-3
(e)	$-\frac{3}{4}$	3π	$\frac{\pi}{2}$	-2

11. Sketch each pair of functions on the same axes. Follow the pattern to generalize $y = \sin k\theta$ in terms of the cosine function.

(a) $y = \sin \theta$ and $y = \cos\left(\theta - \frac{\pi}{2}\right)$, $0 \leq \theta \leq 2\pi$

(b) $y = \sin 2\theta$ and $y = \cos 2\left(\theta - \frac{\pi}{2}\right)$, $0 \leq \theta \leq 2\pi$

(c) $y = \sin 3\theta$ and $y = \cos 3\left(\theta - \frac{\pi}{2}\right)$, $0 \leq \theta \leq 2\pi$

12. For each part of question 10, rewrite the sine function as a cosine function.

13. A skyscraper sways 55 cm back and forth from "the vertical" during high winds. At $t = 5$ s, the building is 55 cm to the right of vertical. The building sways back to the vertical and, at $t = 35$ s, the building sways 55 cm to the left of the vertical. Write an equation that models the motion of the building in terms of time.

14. **Application:** In the "land of the midnight sun," it is daylight all the time during the summer. The first coordinate is the hour of the day. The second coordinate is the angle of elevation of the sun, in degrees, above the horizon at a location in Canada's Far North.

(00:00, 38.59), (01:00, 41.49), (02:00, 42.95), (03:00, 42.75), (04:00, 40.93), (05:00, 37.73), (06:00, 33.51), (07:00, 28.67), (08:00, 23.56), (09:00, 18.52), (10:00, 13.82), (11:00, 9.73), (12:00, 6.46), (13:00, 4.22), (14:00, 3.13), (15:00, 3.26), (16:00, 4.61), (17:00, 7.09), (18:00, 10.55), (19:00, 14.80), (20:00, 19.59), (21:00, 24.67), (22:00, 29.74), (23:00, 34.47), (24:00, 38.48)

(a) Draw a scatter plot of the data and the curve of best fit.

(b) What type of model describes the graph?

(c) Write an equation to model the situation. Describe the constants and the variables in the context of this problem. What restrictions must be placed on the domain?

(d) How could you use the model to calculate the elevation of the sun at 02:00 for the given location?

(e) When is the elevation of the sun above the horizon 30°?

15. The table shows the average monthly high temperature for one year in Kapuskasing.

Time (months)	J	F	M	A	M	J	J	A	S	O	N	D
Temperature (°C)	−18.6	−16.3	−9.1	0.4	8.5	13.8	17.0	15.4	10.3	4.4	−4.3	−14.8

Source: Environment Canada.

(a) Draw a scatter plot of the data and the curve of best fit. Let January be month 0.

(b) What type of model describes the graph?

(c) Write an equation to model the situation. Describe the constants and the variables in the context of this problem.

(d) Use a calculator to graph the equation. Compare the graph and the scatter plot.

(e) What is the average monthly temperature for the 38th month?

16. The depth of water in a harbour on the Bay Fundy that faces the ocean changes each hour, as shown.

Time (h)	00:00	01:00	02:00	03:00	04:00	05:00	06:00	07:00	08:00	09:00	10:00	11:00	12:00
Depth (m)	5.5	6.3	8.5	11.5	14.5	16.7	17.5	16.7	14.5	11.5	8.5	6.3	5.5

(a) Graph the data and determine an equation that models the situation.

(b) Verify the graph using a graphing calculator.

(c) Use the equation to determine the depth of water at 10:30. Verify your answer using the graph.

(d) When is the water 7 m deep?

17. The table shows the velocity of air in litres per second of Nicole's breathing while she is at rest.

Time (s)	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2	2.25	2.5	2.75	3
Velocity (L/s)	0	0.22	0.45	0.61	0.75	0.82	0.85	0.83	0.74	0.61	0.43	0.23	0

(a) Explain why breathing is an example of a periodic function.

(b) Graph the data and determine an equation that models the situation.

(c) Use a graphing calculator to draw the scatter plot of the data. Enter your equation into the equation editor and graph. Comment on the closeness of fit between the scatter plot and the graph.

(d) For $t = 6$ s, what is the velocity of Nicole's breathing? Verify your answer using an alternative method.

(e) How many seconds have passed when the velocity is 0.5 L/s?

18. **Communication:** The table shows the average monthly temperature for Athens, Lisbon, and Moscow. Graph the data to show that temperature is a function of time. Write the equations that model each function. Explain the differences in the amplitude and the vertical shift for each city.

Time (months)	J	F	M	A	M	J	J	A	S	O	N	D
Temperature (°C)												
Athens	12	13	15	19	24	30	33	32	28	23	18	14
Lisbon	13	14	16	18	21	24	26	27	24	21	17	14
Moscow	−9	−6	0	10	19	21	23	22	16	9	1	−4

19. The maximum height of a Ferris wheel is 35 m. The wheel takes 2 min to make one revolution. Passengers board the Ferris wheel 2 m above the ground at the bottom of its rotation.

(a) Write an equation to represent the position of a passenger at any time, t, in seconds.

(b) How high is the passenger after 45 s?

(c) The ride lasts for 4 min. When will the passenger be at the maximum height during this ride?

20. **Check Your Understanding:** Suppose you are given the graph of a sinusoidal function. Explain how you can determine the value of each variable in the corresponding equation $y = a \sin k(\theta + b) + d$.

(a) a (b) k (c) b (d) d

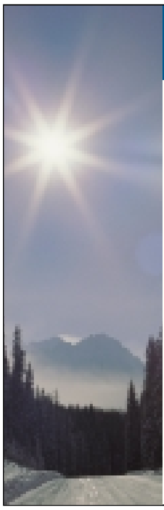
21. **Thinking, Inquiry, Problem Solving:** The diameter of a car's tire is 50 cm. While the car is being driven, the tire picks up a nail.

(a) Model the height of the nail above the ground in terms of the distance the car has travelled since the tire picked up the nail.

(b) How high above the ground will the nail be after the car has travelled 0.5 km?

(c) The nail reaches a height of 10 cm above the ground for the sixth time. How far has the car travelled?

(d) What assumption must you make concerning the driver's habits for the function to give an accurate height?



The Chapter Problem—How Much Daylight?

In this section, you modelled periodic phenomena using trigonometry. Apply what you learned to answer these questions about the Chapter Problem on page 404.

CP11. Write an equation in the form $y = a \sin k(\theta + b) + d$ to model the average number of hours of daylight for the Far North community.

CP12. Use a graphing calculator set to an appropriate scale to graph the equation. Confirm that the graph matches your graphs that you drew for previous sections.

CP13. How many hours of daylight should be expected at month 252?