T5.3 – Vector & Parametric Equations of Lines

IB Math SL1 - Santowski

(A) Review

- Linear Equations have been written as:

- (1) $y = mx + b$ (slope-intercept form)
- (2) $y - y_1 = m(x - x_1)$ (point-slope form)
- (3) $Ax + By + C = 0$ (standard form)

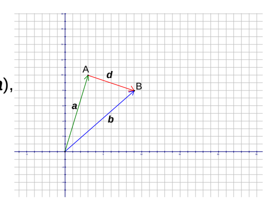
- But now with our look at vectors, there are other ways that we can write equations of lines

(B) Vector Equations of Lines

- To study the vector equation of a line, let's define 2 points as: A(3,10) and B(9,8)
- Draw the line segment AB
- Draw the position vector OA, OB and the vector AB
- Write vector AB as a vector equation
- Write the position vector for point B as an equation

(B) Vector Equations of Lines

- A(3,10) and B(9,8)
- Draw the line segment AB (called ***d***)
- Draw the position vector OA (***a***), OB (***b***) and the vector AB (***d***)
- Write vector AB as a vector equation: ***d*** = (-***a***) + ***b***
- Write the position vector for point B as an equation:
  b = ***a*** + ***d***

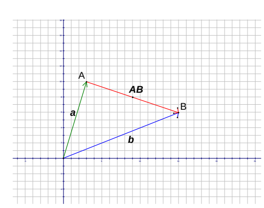


(B) Vector Equations of Lines

- To study the vector equation of a line, let's move point B to a new location: A(3,10) and B(15,6)
- Draw the line segment AB
- Draw the position vector OA, OB and the vector AB ➔ how does this vector compare to ***d*** from our first example?
- Write vector AB as a vector equation
- Write the position vector for point B as an equation

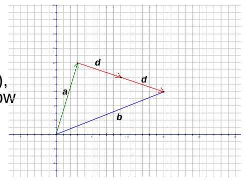
(B) Vector Equations of Lines

- A(3,10) and B(15,6)
- Draw the line segment AB
- Draw the position vector OA (***a***), OB (***b***) and the vector AB ➔ how does this vector compare to ***d*** from our first example?
- Write vector AB as a vector equation ***AB*** = (-***a)*** + ***b***
- Write the position vector for point B as an equation: ***b*** = ***a*** + ***AB***



(B) Vector Equations of Lines

- A(3,10) and B(15,6)
- Draw the line segment AB (called 2***d***)
- Draw the position vector OA (***a***), OB (***b***) and the vector AB ➔ how does this vector compare to ***d*** from our first example ?(2***d***)
- Write vector AB as a vector equation 2***d*** = (-***a***) + ***b***
- Write the position vector for point B as an equation: ***b*** = ***a*** + 2***d***

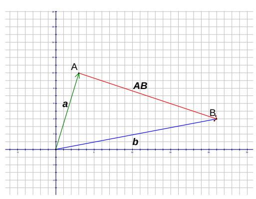


(B) Vector Equations of Lines

- To study the vector equation of a line, let's move point B to a new location: A(3,10) and B(21,4)
- Draw the line segment AB
- Draw the position vector OA, OB and the vector AB ➔ how does this vector compare to ***d*** from our first example?
- Write vector AB as a vector equation
- Write the position vector for point B as an equation

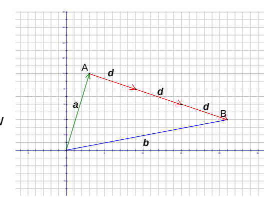
(B) Vector Equations of Lines

- A(3,10) and B(21,4)
- Draw the line segment AB
- Draw the position vector OA (***a***), OB (***b***) and the vector AB ➔ how does this vector compare to ***d*** from our first example
- Write vector AB as a vector equation ***AB*** = (-***a***) + ***b***
- Write the position vector for point B as an equation:
  b = ***a*** + ***AB***



(B) Vector Equations of Lines

- A(3,10) and B(21,4)
- Draw the line segment AB (called 3***d***)
- Draw the position vector OA (***a***), OB (***b***) and the vector AB ➔ how does this vector compare to ***d*** from our first example ?(3***d***)
- Write vector AB as a vector equation 3***d*** = (-***a***) + ***b***
- Write the position vector for point B as an equation: ***b*** = ***a*** + 3***d***



(B) Vector Equations of Lines

- To study the vector equation of a line, let's move point B to a new location: A(3,10) and B(27,2)
- Draw the line segment AB
- Draw the position vector OA, OB and the vector AB ➔ how does this vector compare to ***d*** from our first example?
- Write vector AB as a vector equation
- Write the position vector for point B as an equation

(B) Vector Equations of Lines

- A(3,10) and B(27,2)
- Draw the line segment AB
- Draw the position vector OA (***a***), OB (***b***) and the vector AB ➔ how does this vector compare to ***d*** from our first example
- Write vector AB as a vector equation ***AB*** = (-***a***) + ***b***
- Write the position vector for point B as an equation:
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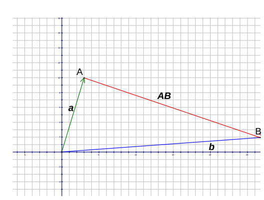
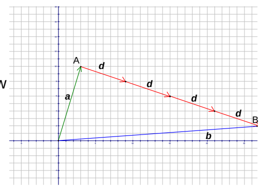


(B) Vector Equations of Lines

- A(3,10) and B(27,2)
- Draw the line segment AB (called 4***d***)
- Draw the position vector OA (***a***), OB (***b***) and the vector AB ➔ how does this vector compare to ***d*** from our first example ?(4***d***)
- Write vector AB as a vector equation 4***d*** = (-***a***) + ***b***
- Write the position vector for point B as an equation: ***b*** = ***a*** + 4***d***

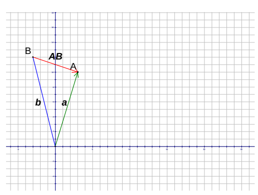


(B) Vector Equations of Lines

- To study the vector equation of a line, let's move point B to a new location: A(3,10) and B(-3,12)
- Draw the line segment AB
- Draw the position vector OA, OB and the vector AB ➔ how does this vector compare to ***d*** from our first example?
- Write vector AB as a vector equation
- Write the position vector for point B as an equation

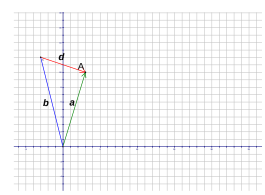
(B) Vector Equations of Lines

- A(3,10) and B(-3,12)
- Draw the line segment AB
- Draw the position vector OA (***a***), OB (***b***) and the vector AB ➔ how does this vector compare to ***d*** from our first example
- Write vector AB as a vector equation -***AB*** = (-***a***) + ***b***
- Write the position vector for point B as an equation:
  b = ***a*** + ***AB***



(B) Vector Equations of Lines

- A(3,10) and B(27,2)
- Draw the line segment AB (called - 1***d***)
- Draw the position vector OA (***a***), OB (***b***) and the vector AB ➔ how does this vector compare to ***d*** from our first example ?(-1***d***)
- Write vector AB as a vector equation -1***d*** = (-***a***) + ***b***
- Write the position vector for point B as an equation: ***b*** = ***a*** - 1***d***



(C) Vector Equations of Lines - Summary

- What do the position vectors of ***OB*** or ***b*** have in common?
- Their points, B, are: B(9,8); B(15,6); B(21,4); B(27,2), B(-3,12) are all ***collinear***
- So the points (3,10), (9,8), (15,6), (21,4), (27,2), (-3,12) all lie on the same line

(C) Vector Equations of Lines - Summary

- Given the example you have just worked through, EXPLAIN the meaning of the vector equation ***b = a + td***
-

(C) Vector Equations of Lines - Summary

- Given the example you have just worked through, EXPLAIN the meaning of the vector equation ***b* = *a* + *td***
- ***b*** represents the position vector of any random point that is on the line
- ***a*** represents the position vector of any random starting point that is on the line
- ***d*** is a direction vector that is parallel (or rather coincident) to the line
- *t* is a parameter and represents the number of multiples of the direction vector that were used in going from the initial point to the second point on the line

(C) Vector Equations of Lines - Summary

- Now, EXPLAIN the similarities of the vector equation ***b* = *a* + *td*** with the linear equation **y = mx + b**
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(C) Vector Equations of Lines - Summary

- Now, EXPLAIN the similarities of the vector equation ***b* = *a* + *td*** with the linear equation ***y* = mx + b**
- ***b*** represents the position vector of any random point that is on the line (**y**)
- ***a*** represents the position vector of any random starting point that is on the line (**b**)
- ***d*** is a direction vector that is parallel (or rather coincident) to the line (**m**)
- *t* is a parameter and represents the number of multiples of the direction vector that were used in going from the initial point to the second point on the line (**x**)

(C) Vector Equations of Lines - Summary

- The linear equation of the line through these points is:
- m = (8 – 10) / (9 – 3) = -2 / 6 = -1/3
- So the linear equation is:
- (1) y – 10 = -1/3(x – 3)
- (2) y = -1/3x + 11
- (3) 0 = x + 3y - 33

(C) Vector Equations of Lines - Summary

- So how would we write a VECTOR equation in the form of ***b* = *a* + *td*** for this line??
- The direction vector, ***d***, (slope) is clearly (6,-2)
- Our original position vector, ***a***, (b) is clearly (3,10)
- So then we have a "slope" and an "initial point", so our equation must be **(x,y) = (3,10) + *t*(6,-2)**
- Then, in "matrix" notation ➔ $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix} + t\begin{bmatrix} 6 \\ -2 \end{bmatrix}$

(D) Parametric Equations of Lines

- Our vector equation of the line was $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix} + t\begin{bmatrix} 6 \\ -2 \end{bmatrix}$
- We can re-write the equation in parametric form as follows:
- (1) note the various x-coordinates used in this investigation (3,9,15,21,27,….)
- ➔ what pattern exists?
- ➔ how can we write an expression for this number pattern?
- (2) note the various y-coordinates used in this investigation (10,8,6,4,2,….)
- ➔ what pattern exists?
- ➔ how can we write an expression for this number pattern?

(D) Parametric Equations of Lines

- Our vector equation of the line was $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix} + t\begin{bmatrix} 6 \\ -2 \end{bmatrix}$
- We can re-write the equation in parametric form as follows:
- (1) note the various x-coordinates used in this investigation (3,9,15,21,27,….) → what pattern exists? **→ starting from 3, increments by 6 every time**
- how can we write an expression for this number pattern?
  x(t) = 3 + 6t where t is the number of increments
- (2) note the various y-coordinates used in this investigation (10,8,6,4,2,….) → what pattern exists? **→ starting from 10, increments by -2 every time**
- → how can we write an expression for this number pattern?
  y(t) = 10 + -2t where t is the number of increments

(E) Vector Equations of the Line - Summary

- Our vector equation of the line was $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix} + t\begin{bmatrix} 6 \\ -2 \end{bmatrix}$
- We can re-write the equation in parametric form as follows:

$$x(t) = 3 + 6t$$
$$y(t) = 10 - 2t$$

(E) Vector Equations of the Line - Summary

- And with our parametric equation, we can isolate the parameter, t:

$$x(t) = 3 + 6t \quad \Rightarrow \quad t = \frac{x-3}{6}$$
$$y(t) = 10 - 2t \quad \Rightarrow \quad t = \frac{10-y}{2}$$

- So we can equate our parameter and create a symmetrical equation

$$t = \frac{x-3}{6} = \frac{10-y}{2}$$

(F) Examples

- (1) Consider the line through A = (2,3) and B = (5,-1). Write the vector and parametric equations for the line AB. Does the point (-31,47) lie on line AB?
- (2) Find the vector equation and parametric equations for the line through the point (−2, 4, 10) and parallel to the vector (3, 1,−8).
- (3) Find the vector equation and parametric equations of the line passing through P = (1, 0, –1) and Q = (–2, 1, 3)

(F) Examples

- (4) Consider the two vector descriptions:
  < x,y > = < 2,3 > + t < 1,-4 > and
  < x,y > = < -1,15 > + t < -2,8 >.

Do they describe the same line? Justify your answer thoroughly.

- (5) Consider the line described by
  < x,y > = < 5,0 > + t < 1,-2 >
- a. Show that the point (8,3) does not lie on that line.
- b. Give a vector description of the line parallel to that line through (8,3) .
- c. Give a vector description of the line perpendicular to that line
- through (8,3) .

Example

Find the parametric equations of the line passing through the point P = (1, 3, –2) and parallel to the line with the vector equation **r** = [2t, 1+ t, 4 – 3t]

Example

Find the parametric equations of the line passing through the point P = (1, 3, –2) and parallel to the line with the vector equation: **r** = [2t, 1+ t, 4 – 3t]

Answer: We put the vector equation in the form:

r = **OP** + t**v** = [0, 1, 4] + t[2, 1, – 3],

thus getting the direction vector **v** = [2, 1, – 3].

Finally, we write the parametric equations:

x = 1 + 2t, y = 3 + t, z = –2 – 3t.

Example

Ex. Consider the line that passes through (5,1,3) and is parallel to **i** + 4**j** – 2**k**.

a) Find the vector and parametric equations of the line

b) Identify two other points on the line.

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a) Find the vector and parametric equations of the line

b) Identify two other points on the line.

(F) Examples

1 The points A and B have position vectors $\begin{pmatrix}1\\6\\4\end{pmatrix}$ and $\begin{pmatrix}5\\0\\-6\end{pmatrix}$ respectively, relative to a fixed origin.

a Find, in vector form, an equation of the line l which passes through A and B. **(2)**

The line m has equation

$$\mathbf{r} = \begin{pmatrix}5\\-5\\3\end{pmatrix} + t\begin{pmatrix}1\\-4\\2\end{pmatrix}.$$

Given that lines l and m intersect at the point C,

b find the position vector of C, **(4)**

c show that C is the mid-point of AB. **(2)**

(F) Examples

2 Relative to a fixed origin, the points P and Q have position vectors $\begin{pmatrix}5\\-2\\2\end{pmatrix}$ and $\begin{pmatrix}3\\1\\0\end{pmatrix}$ respectively.

a Find, in vector form, an equation of the line L_1 which passes through P and Q. **(2)**

The line L_2 has equation

$$\mathbf{r} = \begin{pmatrix}4\\6\\-1\end{pmatrix} + \mu\begin{pmatrix}5\\-1\\3\end{pmatrix}.$$

b Show that lines L_1 and L_2 intersect and find the position vector of their point of intersection. **(5)**

c Find, in degrees to 1 decimal place, the acute angle between lines L_1 and L_2. **(3)**

(G) Homework

- HW - day 1
- Ex 17A #1ac, 2, 3ab, 4ab;
- Ex 17E #3, 4
- HW – day 2
- Ex 17F #1, 2, 3ad;
- Ex 17G #1cde;
- IB Packet #6