T4.3 - Inverse of Matrices & Determinants

IB Math SL - Santowski

(A) Review

- - at this stage of studying matrices, we know how to add, subtract and multiply matrices
- i.e. if $A = \begin{bmatrix} 2 & -3 \\ -5 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1/2 & 1 \\ 3 & -1 \end{bmatrix}$
- Then evaluate:
- (a) A + B
- (b) -3A
- (c) BA
- (d) B – A
- (e) AB

(B) Review of Real Numbers

- if we divide 5 by 8 (i.e. 5/8), we could rearrange and look at division as nothing more than simple multiplication
- ➔ thus 5/8 = 5 x 1/8 = 5 x 8^{-1}
- ➔ so in a way, we would never have to perform division as long as we simply multiply by the inverse (or reciprocal)
- One other note about this inverse of a number ➔ a number and its inverse (its reciprocal) have the property that (n) x (n^{-1}) = 1
- - i.e. (8) (8^{-1}) = (8) (1/8) = (8/8) = 1
- So how does this relate to DIVISION of MATRICES????

(C) Strategy for "Dividing" Matrices

- So how does "multiplicative inverses" relate to DIVISION of MATRICES????
- If a number and its inverse (its reciprocal) have the property that (n) x (n^{-1}) = 1
- Then ….

(C) Strategy for "Dividing" Matrices

- So how does "multiplicative inverses" relate to DIVISION of MATRICES????
- If a number and its inverse (its reciprocal) have the property that (n) x (n^{-1}) = 1
- Then …. a matrix and its "inverse" should have the property that ***B*** x ***B*** $^{-1}$ = 1

(C) Strategy for "Dividing" Matrices

- So …. a matrix and its "inverse" should have the property that ***B*** x ***B*** $^{-1}$ = 1
- Well what is 1 in terms of matrices? ➔ simply the ***<u>identity</u>*** matrix, ***I***
- Thus ***B*** x ***B*** $^{-1}$ = ***I*** $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(D) Inverse Matrices

- Given matrix A, which of the following 4 is the inverse of matrix A?

$$A=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$$

$$B=\begin{bmatrix}\frac{1}{2} & \frac{1}{3}\\ 1 & \frac{1}{2}\end{bmatrix} \qquad C=\begin{bmatrix}-2 & -3\\ -1 & -2\end{bmatrix}$$

$$D=\begin{bmatrix}2 & 1\\ -3 & -2\end{bmatrix} \qquad E=\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}$$

(D) Inverse Matrices

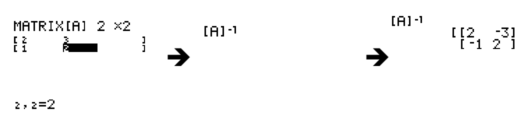
- Solve for x:

$$\text{Let } A=\begin{bmatrix}1 & 0 & -2\\ 4 & 1 & 0\\ 1 & 1 & 7\end{bmatrix}, \text{ then } A^{-1}=\begin{bmatrix}x & -2 & 2\\ -28 & 9 & -8\\ 3 & -1 & 1\end{bmatrix}$$

(E) Terms Associated with Inverse Matrices

- Thus we have 2 new terms that relate to inverse matrices:
- (a) a matrix is ***invertible*** if it has an inverse
- (b) a matrix is ***singular*** if it does NOT have an inverse

(F) Inverse Matrices on TI-83/4

- So we have the basic idea of inverse matrices ➔ how can I use the calculator to find the inverse of a matrix??
- 

(F) Inverse Matrices on TI-83/4

- Use the TI-83/4 to determine the inverse of:

$$A=\begin{bmatrix}2 & 1\\ 4 & 3\end{bmatrix}$$

$$B=\begin{bmatrix}2 & 3\\ 5 & -4\end{bmatrix}$$

(G) Properties of Inverses (and Matrix Multiplication)

- Is multiplication with real numbers commutative (is $ab = ba$)?
- Is matrix multiplication commutative
  - ➔ Is ***AB*** = ***BA***? (use TI-84 to investigate)
- Is $\boldsymbol{A} \times \boldsymbol{A}^{-1} = \boldsymbol{A}^{-1} \times \boldsymbol{A} = I$? (use TI-84 to investigate)

(G) Properties of Inverses (and Matrix Multiplication)

- Are these "properties" true for (i) real numbers? (ii) matrices??? Use TI-84 to investigate
- Is $(A^{-1})^{-1} = A$???
- Is $(AB)^{-1} = A^{-1}B^{-1}$?

(H) Determining the Inverse of a Matrix

- How can we determine the inverse of a matrix if we DO NOT have access to our calculators?
- (i) Matrix Multiplication
- (ii) Calculating the "determinant"

(H) Determining the Inverse of a Matrix

- Let's use Matrix Multiplication to find the inverse of $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$
- So our matrix will be $A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- And we now have the multiplication $A \times A^{-1} = I$ $\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
- And so using our knowledge of matrix multiplication, we get ➔

(H) Determining the Inverse of a Matrix

- And so using our knowledge of matrix multiplication, we get a system of 4 equations ➔

$$\begin{cases} a + c = 1 \\ b + d = 0 \\ -a + 2c = 0 \\ -b + 2d = 1 \end{cases}$$

- Which we can solve as:

$a + c = 1$ and $-a + 2c = 0 \Rightarrow a = 2c$

so $(2c) + c = 1$

so $c = 1/3$ and $a = 2/3$

$-b + 2d = 1$ and $b + d = 0 \Rightarrow -b = d$

so $(d) + 2d = 1$

so $d = 1/3$ and $b = -1/3$

(H) Determining the Inverse of a Matrix

- So if $A = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$
- So our matrix will be $A^{-1} = \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}$

(H) Determining the Inverse of a Matrix

- How can we determine the inverse of a matrix if we DO NOT have access to our calculators?
- (ii) Calculating the "determinant"
- So Method #2 involved something called a "determinant" ➔ which means …..??

(I) Determinants → An Investigation

- Use your TI-83/4 to determine the following products:

$$\begin{bmatrix}3 & 1\\5 & 2\end{bmatrix}\times\begin{bmatrix}2 & -1\\-5 & 3\end{bmatrix}=\begin{bmatrix} & \\ & \end{bmatrix}=$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\times\begin{bmatrix}4 & -2\\-3 & 1\end{bmatrix}=\begin{bmatrix} & \\ & \end{bmatrix}=\begin{bmatrix} & \\ & \end{bmatrix}=$$

$$\begin{bmatrix}5 & 11\\-2 & 3\end{bmatrix}\times\begin{bmatrix}3 & -11\\2 & 5\end{bmatrix}=\begin{bmatrix} & \\ & \end{bmatrix}=\begin{bmatrix} & \\ & \end{bmatrix}=$$

(I) Determinants → An Investigation

- Use your TI-83/4 to determine the following products:

$$\begin{bmatrix}3 & 1\\5 & 2\end{bmatrix}\times\begin{bmatrix}2 & -1\\-5 & 3\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\times\begin{bmatrix}4 & -2\\-3 & 1\end{bmatrix}=\begin{bmatrix}-2 & 0\\0 & -2\end{bmatrix}=-2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=-2I$$

$$\begin{bmatrix}5 & 11\\-2 & 3\end{bmatrix}\times\begin{bmatrix}3 & -11\\2 & 5\end{bmatrix}=\begin{bmatrix}37 & 0\\0 & 37\end{bmatrix}=37\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=37I$$

(I) Determinants → An Investigation

- Now carefully look at the 2 matrices you multiplied and observe a pattern ????

$$\begin{bmatrix}3 & 1\\5 & 2\end{bmatrix}\times\begin{bmatrix}2 & -1\\-5 & 3\end{bmatrix}=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=I$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}\times\begin{bmatrix}4 & -2\\-3 & 1\end{bmatrix}=\begin{bmatrix}-2 & 0\\0 & -2\end{bmatrix}=-2\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=-2I$$

$$\begin{bmatrix}5 & 11\\-2 & 3\end{bmatrix}\times\begin{bmatrix}3 & -11\\2 & 5\end{bmatrix}=\begin{bmatrix}37 & 0\\0 & 37\end{bmatrix}=37\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}=37I$$

(I) Determinants → An Investigation

- Now carefully look at the 2 matrices you multiplied and observe a pattern ????

$$\begin{bmatrix}a & b\\c & d\end{bmatrix}\times\begin{bmatrix}?? & ??\\?? & ??\end{bmatrix}=\begin{bmatrix}?? & 0\\0 & ??\end{bmatrix}=?I$$

(I) Determinants → An Investigation

- Now PROVE your pattern holds true for all values of *a*, *b*, *c*, *d* ….

(I) Determinants → An Investigation

- Now PROVE your pattern holds true for all values of *a*, *b*, *c*, *d* ….

- $$\begin{bmatrix}a & b\\c & d\end{bmatrix}\times\begin{bmatrix}d & -b\\-c & a\end{bmatrix}$$

$$=\begin{bmatrix}ad-bc & -ab+ab\\cd-cd & -bc+ad\end{bmatrix}$$

$$=\begin{bmatrix}ad-bc & 0\\0 & -bc+ad\end{bmatrix}$$

$$=(ad-bc)\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

$$=(ad-bc)I$$

(I) Determinants ➔ An Investigation

- So to summarize:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (ad - bc) \times \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

OR

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \frac{1}{(ad - bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(I) Determinants ➔ An Investigation

- then we see that from our original matrix, the value (*ad-bc*) has special significance, in that its value ***determines*** whether or not matrix *A* can be inverted
- -if *ad* - *bc* does not equal 0, matrix *A* would be called "invertible"
- - i.e. if *ad* - *bc* = 0, then matrix *A* cannot be inverted and we call it a ***singular matrix***
- - the value *ad* - *bc* has a special name ➔ it will be called the ***determinant*** of matrix *A* and has the notation det***A*** or |***A***|

(I) Determinants ➔ An Investigation

- So if ***A*** is invertible then

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \text{then} \quad A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{where } |A| = ad - bc$$

(J) Examples

- ex 1. Find the determinant of the following matrices and hence find their inverses:
- Verify using TI-83/4

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 \\ 5 & -4 \end{bmatrix}$$

(J) Examples

- ex 2. Find the determinant of the following matrices and hence find their inverses:
- Verify using TI-83/4

$$A = \begin{bmatrix} 5 & 3 \\ 5 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 3 & 4 \\ 1 & -2 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 3 \\ 7 & 8 \end{bmatrix} \quad D = \begin{bmatrix} 5 & -2 \\ 3 & 6 \end{bmatrix}$$

(J) Examples

- Prove whether the following statements are true or false for 2 by 2 matrices. Remember that a counterexample establishes that a statement is false.
- In general, you may NOT assume that a statement is true for all matrices because it is true for 2 by 2 matrices, but for the examples in this question, those that are true for 2 by 2 matrices are true for all matrices if the dimensions allow the operations to be performed.

- Questions:

(a) $\det(AB) = \det(A)\det(B)$

(b) $\det(A^{-1}) = \frac{1}{\det(A)}$

(c) $\det(A + B) = \det(A) + \det(B)$

(d) $\det(A^T) = \det(A)$

(L) Homework

- HW –

- Ex 14H #2ad, 8acf;
- Ex 14I #1a, 3ab, 4b, 7;
- Ex 14K #2a;
- Ex 14L #5a, 8;

- IB Packet #2, 7