T4.2 – Multiplication of Matrices

IB Math SL - Santowski

(A) Matrix Multiplication by Scalars - Review

- Recall our “Breakfast” Matrix

- Pancakes: 2 cups baking mix, 2 eggs, and 1 cup milk.
- Biscuits: 2 ¼ cups baking mix and ¾ cups milk.
- Waffles: 2 cups baking mix, 1 egg, 1⅓ cups milk, and 2 tablespoons vegetable oil.

- Let's write this in the form of a labeled matrix so that it is easier to read.

(A) Matrix Multiplication by Scalars - Review

- Recall our “Breakfast” Matrix

- Pancakes: 2 cups baking mix, 2 eggs, and 1 cup milk.
- Biscuits: 2 ¼ cups baking mix and ¾ cups milk.
- Waffles: 2 cups baking mix, 1⅓ egg, cups milk, and 2 tablespoons vegetable oil.

- $$R = \begin{bmatrix} & Bm & E & M & O \\ P & 2 & 2 & 1 & 0 \\ B & 2\frac{1}{4} & 0 & \frac{3}{4} & 0 \\ W & 2 & 1 & 1\frac{1}{3} & 2 \end{bmatrix} \rightarrow R = \begin{bmatrix} 2 & 2 & 1 & 0 \\ 2\frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 2 & 1 & 1\frac{1}{3} & 2 \end{bmatrix}$$

(B) Matrix Multiplication

- Now, if we want to feed 3 people pancakes, 12 people biscuits, and 9 people waffles, how much baking mix will we need?

- We need to make one batch of pancakes, 4 batches of biscuits, and 3 batches of waffles.

$$R = \begin{bmatrix} & Bm & E & M & O \\ P & 2 & 2 & 1 & 0 \\ B & 2\frac{1}{4} & 0 & \frac{3}{4} & 0 \\ W & 2 & 1 & 1\frac{1}{3} & 2 \end{bmatrix}$$

(B) Matrix Multiplication

- Now, if we want to feed 3 people pancakes, 12 people biscuits, and 9 people waffles, how much baking mix will we need?
- We need to make one batch of pancakes, 4 batches of biscuits, and 3 batches of waffles.

- We can work it out algebraically as

$$(1\times 2)+\left(4\times 2\frac{1}{4}\right)+(2\times 3)=17$$

- But how do we “create” the same solution as matrix multiplication??

$$R = \begin{bmatrix} & Bm & E & M & O \\ P & 2 & 2 & 1 & 0 \\ B & 2\frac{1}{4} & 0 & \frac{3}{4} & 0 \\ W & 2 & 1 & 1\frac{1}{3} & 2 \end{bmatrix}$$

(B) Matrix Multiplication

- But how do we “create” the same solution using matrix multiplication?? ➔ we need to create a “meaning” to our matrices again, so …..

- Let ***n*** = “numbers” matrix (as a row matrix) $n = \begin{matrix} P & B & W \\ [1 & 4 & 3] \end{matrix}$

- Let ***I*** = “ingredient” matrix (as a column matrix) $I = \begin{matrix} P \\ B \\ W \end{matrix} \begin{bmatrix} 2 \\ 2\frac{1}{4} \\ 2 \end{bmatrix}$

- So our total required batter would be represented by the matrix multiplication ➔ ***n*** x ***I***

(B) Matrix Multiplication

- So our total required batter would be represented by the matrix multiplication ➔ ***n*** x ***I***
- So now to show the matrix multiplication:

$$n \times I = \begin{bmatrix}1 & 4 & 3\end{bmatrix} \times \begin{bmatrix}2 \\ 2\frac{1}{4} \\ 2\end{bmatrix} = \left[(1\times2)+\left(4\times2\frac{1}{4}\right)+(3\times2)\right] = [17]$$

- So the process ➔ each element in the row matrix was multiplied by a corresponding element in the column matrix.
- So the process ➔ each of the products were summed together
- So a note in passing ➔ *n* was a 1 by 3 matrix and ***I*** was a 3 by 1 matrix and our product was a 1 by 1 matrix …

(B) Matrix Multiplication

- Now to expand upon our example ➔ we have only worked out how much baking mix I need ……
- What would we do if we wanted to know how much of each ingredient we need for 1 batch of pancakes, 4 batches of biscuits, and 3 batches of waffles?

(B) Matrix Multiplication

- Now to expand upon our example ➔ we have only worked out how much baking mix I need ……
- What would we do if we wanted to know how much of each ingredient we need for 1 batch of pancakes, 4 batches of biscuits, and 3 batches of waffles?
- Well, we would run through 3 more separate multiplications, one for each ingredient (i.e. milk) ➔ OR …..

$$n \times I = \begin{bmatrix}1 & 4 & 3\end{bmatrix} \times \begin{bmatrix}1 \\ \frac{3}{4} \\ 1\frac{1}{3}\end{bmatrix} = \left[(1\times1)+\left(4\times\frac{3}{4}\right)+\left(3\times1\frac{1}{3}\right)\right] = [8]$$

(B) Matrix Multiplication

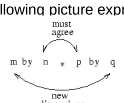
- Why not use our ingredient matrix?????

$$n \times R = \begin{bmatrix}1 & 4 & 3\end{bmatrix} \times \begin{bmatrix}2 & 2 & 1 & 0 \\ 2\frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 2 & 1 & 1\frac{1}{3} & 2\end{bmatrix} = \begin{bmatrix}17 & 5 & 8 & 6\end{bmatrix}$$

$$n \times R = \begin{matrix} Bm & E & M & O \\ [17 & 5 & 8 & 6] \end{matrix}$$

(B) Matrix Multiplication

- So, our example suggests a method for doing matrix multiplication and some conditions for multiplication …..
- We multiplied a 1 by 3 matrix by a 3 by 4 matrix and got a 1 by 4 matrix
- The middle numbers (columns of one matrix and rows of the second matrix) must be the same (like the threes were in this case). The resulting matrix will always have the dimensions of the outside numbers (1 by 4 in this case) when multiplication is defined.
- The following picture expresses the requirements on the dimensions:



$$n \times R = \begin{bmatrix}1 & 4 & 3\end{bmatrix} \times \begin{bmatrix}2 & 2 & 1 & 0 \\ 2\frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 2 & 1 & 1\frac{1}{3} & 2\end{bmatrix} = \begin{bmatrix}17 & 5 & 8 & 6\end{bmatrix}$$

$$n \times R = \begin{matrix} Bm & E & M & O \\ [17 & 5 & 8 & 6] \end{matrix}$$

(B) Matrix Multiplication

- And to wrap up our "breakie" matrix question ➔ we also know our calorie intake and costs of each of our ingredients ➔ so……

$$C = \begin{matrix} Bm \\ E \\ M \\ O \end{matrix}\overset{\text{cal}}{\begin{bmatrix}510 \\ 70 \\ 90 \\ 120\end{bmatrix}} \quad \text{and} \quad \$ = \begin{matrix} Bm \\ E \\ M \\ O \end{matrix}\overset{\text{cost}}{\begin{bmatrix}0.17 \\ 0.08 \\ 0.13 \\ 0.04\end{bmatrix}}$$

(B) Matrix Multiplication

- ➔ so we could work out our costs and calories for our pancakes, biscuits & waffles

$$R \times F = \begin{bmatrix} 2 & 2 & 1 & 0 \\ 2\frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 2 & 1 & 1\frac{1}{3} & 2 \end{bmatrix} \times \begin{bmatrix} 0.17 & 510 \\ 0.08 & 70 \\ 0.13 & 90 \\ 0.04 & 120 \end{bmatrix} = [?]$$

(B) Matrix Multiplication

- ➔ so we could work out our costs and calories for our pancakes, biscuits & waffles

$$R \times F = \begin{bmatrix} 2 & 2 & 1 & 0 \\ 2\frac{1}{4} & 0 & \frac{3}{4} & 0 \\ 2 & 1 & 1\frac{1}{3} & 2 \end{bmatrix} \times \begin{bmatrix} 0.17 & 510 \\ 0.08 & 70 \\ 0.13 & 90 \\ 0.04 & 120 \end{bmatrix} = \begin{bmatrix} 0.63 & 1250 \\ 0.48 & 1215 \\ 0.67\frac{1}{3} & 1450 \end{bmatrix}$$

(C) Key Terms for Matrices

- We learned in the last lesson that there is a matrix version of the addition property of zero.
- There is also a matrix version of the multiplication property of one.
- The real number version tells us that if *a* is a real number, then *a**1 = 1**a* = *a*.
- The matrix version of this property states that if *A* is a square matrix, then *A***I* = *I***A* = *A*, where *I* is the **identity matrix** of the same dimensions as *A*.
- **Definition ➔** An identity matrix is a square matrix with ones along the main diagonal and zeros elsewhere.

(C) Key Terms for Matrices

- **Definition ➔** An identity matrix is a square matrix with ones along the main diagonal and zeros elsewhere.

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- So, in matrix multiplication ➔ **A x I = I x A = A**

(D) Examples for Practice

- Multiply the following matrices:

(a) $\begin{bmatrix} 2 & 0 & -1 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 5 & -7 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & 0 & 0 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & -1 & -2 \end{bmatrix} \times \begin{bmatrix} 4 & 2 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

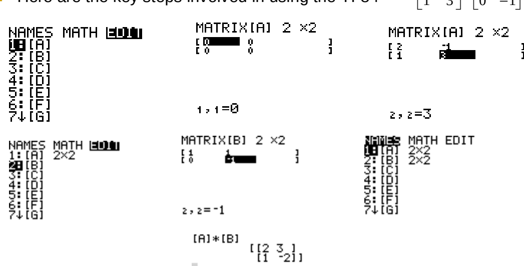
(d) $\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$

(e) $\begin{bmatrix} 3 & -1 & -2 \\ 2 & -2 & -1 \end{bmatrix} \times \begin{bmatrix} 4 & 2 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 4 & 6 \end{bmatrix}$

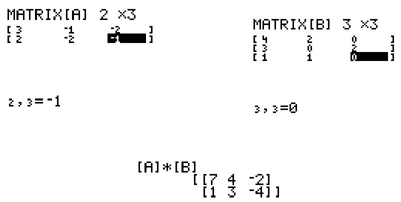
(E) Examples for Practice – TI-84

- Here are the key steps involved in using the TI-84 (a) $\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$



(E) Examples for Practice – TI-84

- Here are the key steps involved in using the TI-84 (a) $\begin{bmatrix} 3 & -1 & -2 \\ 2 & -2 & -1 \end{bmatrix} \times \begin{bmatrix} 4 & 2 & 0 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix}$



(F) Properties of Matrix Multiplication

- Now we pass from the concrete to the abstract → What properties are true of matrix multiplication where we simply have a matrix (wherein we know or don't know what elements are within)
- Asked in an alternative sense → what are the general properties of multiplication (say of real numbers) in the first place???

(F) Properties of Matrix Multiplication

Ordinary Algebra with Real numbers	**Matrix Algebra**
if *a* and *b* are real numbers, so is the product *ab*	if **A** and **B** are matrices, so is the product **AB**
ab = *ba* for all *a*,*b*	in general, **AB ≠ BA**
*a*0 = 0*a* = 0 for all *a*	**A**0 = 0**A** = 0 for all **A** where 0 is the zero matrix
a(*b* + *c*) = *ab* + *ac*	**A**(**B** + **C**) = **AB** + **AC**
a x 1 = 1 x *a* = *a*	**AI** = **IA** = **A** where I is called an identity matrix and A is a square matrix
a^n exists for all *a* ≥ 0	$\mathbf{A}^n$ for {n E *I* | n ≥ 2} and **A** is a square matrix

(F) Properties of Matrix Multiplication

- This is a good place to use your calculator if it handles matrices. Do enough examples of each to convince yourself of your answer to each question
- (1) Does *AB* = *BA* for all *B* for which matrix multiplication is defined if $A = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$?
- (2) In general, does *AB* = *BA*?
- (3) Does *A*(*BC*) = (*AB*)*C*?
- (4) Does *A*(*B* + *C*) = *AB* + *AC*?

(F) Properties of Matrix Multiplication

- This is a good place to use your calculator if it handles matrices. Do enough examples of each to convince yourself of your answer to each question
- (5) Does $(AB)^T = B^T A^T$?
- (6) Does *A* - *B* = -(*B* - *A*)?
- (7) For real numbers, if *ab* = 0, we know that either *a* or *b* must be zero. Is it true that *AB* = 0 implies that *A* or *B* is a zero matrix?
- (8) Are $A^T A$ and AA^T always symmetric?

(H) Homework

HW:

Ex 14E.2 #1 - 4;
Ex 14F #1bd, 3;
Ex 14G #5 - 7, RS
Ex 14A #1ghk, RS
Ex 14E #3bcd;
IB Packet #3, 6