T.3.2 - Trigonometric Identities

IB Math SL - Santowski

(A) Review of Equations

- An equation is an algebraic statement that is true for only several values of the variable
- The linear equation 5 = 2x – 3 is only true for the x value of 4
- The quadratic equation $0 = x^2 - x - 6$ is true only for x = -2 and x = 3 (i.e. 0 = (x – 3)(x + 2))
- The trig equation sin(θ) = 1 is true for several values like 90°, 450° , -270°, etc…
- The reciprocal equation 2 = 1/x is true only for x = ½
- The root equation 4 = √x is true for one value of x = 16

(B) Introduction to Identities

- Now imagine an equation like 2x + 2 = 2(x + 1) and we ask ourselves the same question ➔ for what values of x is it true?
- We can actually see very quickly that the right side of the equation expands to 2x + 2, so in reality we have an equation like 2x + 2 = 2x + 2
- But the question remains ➔ for what values of x is the equation true??
- Since both sides are identical, it turns out that the equation is true for **ANY** value of x we care to substitute!
- So we simply assign a slightly different name to these special equations ➔ we call them IDENTITIES because they are true for ALL values of the variable!

(B) Introduction to Identities

- The key to working with identities is to show that the two "sides" are identical expressions ➔ HOWEVER, it may not always be obvious that the two sides of an "equation" or rather an identity are actually the same expression
- For example, $4(x - 2) = (x - 2)(x + 2) - (x - 2)^2$
- Is this an identity (true for ALL values of x) or simply an equation (true for one or several values of x)???
- The answer lies with our mastery of fundamental algebra skills like expanding and factoring ➔ so in this case, we can perform some algebraic simplification on the right side of this equation
- $RS = (x^2 - 4) - (x^2 - 4x + 4)$
- RS = -4 + 4x – 4
- RS = 4x – 8
- RS = 4(x – 2)
- RS = LS
- So yes, this is an identity since we have shown that the sides of the "equation" are actually the same expression

(C) Basic Trigonometric Identities

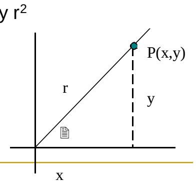
- Recall our definitions for sin(θ) = o/h, cos(θ) = a/h and tan(θ) = o/a
- So now one trig identity can be introduced ➔if we take sin(θ) and divide by cos(θ), what do we get?
- $\frac{\sin(\theta)}{\cos(\theta)} = \frac{o/h}{a/h} = \frac{o}{a} = \tan(\theta)$
- So the tan ratio is actually a quotient of the sine ratio divided by the cosine ratio

(C) Basic Trigonometric Identities

- So the tan ratio is actually a quotient of the sine ratio divided by the cosine ratio
- We can demonstrate this in several ways ➔ we can substitute any value for θ into this equation and we should notice that both sides always equal the same number
- Or we can graph f(θ) = sin(θ)/cos(θ) as well as f(θ) = tan(θ) and we would notice that the graphs were identical
- This identity is called the QUOTIENT identity

(C) Basic Trigonometric Identities

- Another key identity is called the Pythagorean identity
- In this case, since we have a right triangle, we apply the Pythagorean formula and get $x^2 + y^2 = r^2$
- Now we simply divide both sides by r^2



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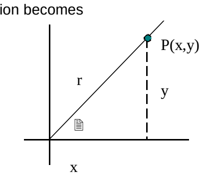
(C) Basic Trigonometric Identities

- Now we simply divide both sides by r^2
- and we get $x^2/r^2 + y^2/r^2 = r^2/r^2$
- Upon simplifying, $(x/r)^2 + (y/r)^2 = 1$
- But $x/r = \cos(\theta)$ and $y/r = \sin(\theta)$ so our equation becomes
- $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$
- Or rather $\cos^2(\theta) + \sin^2(\theta) = 1$
- Which again can be demonstrated by
- substitution or by graphing



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(D) Solving Trig Equations with Substitutions ➔ Identities

- Solve $\tan\theta \times \cos\theta - 1 = 0 \quad \text{for } -2\pi \le \theta \le 2\pi$

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(D) Solving Trig Equations with Substitutions ➔ Identities

- Solve $\tan\theta \times \cos\theta - 1 = 0 \quad \text{for } -2\pi \le \theta \le 2\pi$
- But $\tan x \cos x = \dfrac{\sin x}{\cos x}\cos x$

  So $\tan x \cos x = \sin x$
- So we make a substitution and simplify our equation ➔ $\sin\theta - 1 = 0 \quad \text{for } -2\pi \le \theta \le 2\pi$

$$\theta = -\frac{3\pi}{2}, \frac{\pi}{2}$$

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(E) Examples

- Solve $\dfrac{\sin^2 x}{1 - \cos x} = 2 \quad \text{for } -2\pi \le x \le 2\pi$

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(E) Examples

- Solve $\dfrac{\sin^2 x}{1 - \cos x} = 2 \quad \text{for } -2\pi \le x \le 2\pi$
- Now, one option is:

$$\frac{\sin^2 x}{1 - \cos x} = 2 \quad \text{for } -2\pi \le x \le 2\pi$$
$$\frac{1 - \cos^2 x}{1 - \cos x} = 2$$
$$\frac{(1 - \cos x)(1 + \cos x)}{1 - \cos x} = 2$$
$$1 + \cos x = 2$$
$$\therefore \cos x = -1$$
$$x = \pm\pi$$

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(E) Examples

- Solve the following

(a) $\sin x + 1 - 2\cos^2 x = 0 \quad \text{for } -2\pi \le x \le 2\pi$

(b) $1 - \sin x = 2\cos^2 x \quad \text{for } -2\pi \le x \le 2\pi$

(c) $\frac{1}{\cos x} - \sin x \tan x = -\frac{1}{\sqrt{2}} \quad \text{for } -2\pi \le x \le 2\pi$

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(F) Example

- Since 1- $\cos^2 x = \sin^2 x$ is an identity, is

$$\sqrt{1-\cos^2 x} = \sin x$$

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(F) Example

- If sin(x) = 2/3 and x is an angle in the second quadrant, determine the values of the trig ratios for cos(x) and tan(x)
- If tan(x) = 8/15 and x is an angle in the 3rd quadrant, determine the values for sin(x) and for cos(x)

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(G) Simplifying Trig Expressions

- Simplify the following expressions:

$(a)\ 2 - 2\cos^2 x$

$(b)\ \sin^2 x \cos x + \cos^3 x$

$(c)\ (\cos x - \sin x)^2$

$(d)\ \frac{2-2\cos^2 x}{1+\cos x}$

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(F) Homework

- HW
- Ex 13C.1 #2ad, 3bc, 5 (students should also find the value of tan for all exercises);
- Ex 13I # #1de, 2agek, 3ac, 4abfghi, 5a, 6bc

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