T.2.8 – Base e and Natural Logs

IB Math SL1 – Santowski

(A) Introducing Base *e*

- One way to introduce the number *e* is to use compounding as in the following example:
- Take $1000 and let it grow at a rate of 10% p.a. Then determine value of the $1000 after 2 years under the following compounding conditions:
- (i) compounded annually
- $=1000(1 + .1/1)^{(2\times1)}$
- (ii) compounded quarterly
- $=1000(1 + 0.1/4)^{(2\times4)}$
- (iii) compounded daily
- $=1000(1 + 0.1/365)^{(2\times365)}$
- (iv) compounded *n* times per year
- $=1000(1+0.1/n)^{(2\times n)}$

(A) Introducing Base *e*

- So we have the expression $1000(1 + 0.1/n)^{(2\times n)}$
- Now what happens as we increase the number of times we compound per annum ⇒ i.e. $n \to \infty$?? (that is come to the point of compounding continuously)
- So we get a limit:

$$\lim_{n\to\infty}\left(1000\times\left(1+\frac{0.1}{n}\right)^{(2\times n)}\right)$$

(A) Introducing Base *e*

- Now let's rearrange our limit using a simple substitution => let 0.1/n = 1/x
- Therefore, 0.1x = n ==> so then $\lim_{n\to\infty}\left(1000\times\left(1+\frac{0.1}{n}\right)^{(2\times n)}\right)$
- becomes $\lim_{x\to\infty}\left(1000\times\left(1+\frac{1}{x}\right)^{(x\times0.1\times2)}\right)$
- Which simplifies to $1000\times\lim_{x\to\infty}\left(\left(1+\frac{1}{x}\right)^{x}\right)^{0.1\times2}$

(A) Introducing Base *e*

- So we see a special limit occurring:

$$\lim_{x\to\infty}\left(\left(1+\frac{1}{x}\right)^x\right)$$

- We can evaluate the limit a number of ways ==> graphing or a table of values.
- In either case, $\lim_{x\to\infty}\left(\left(1+\frac{1}{x}\right)^x\right)=e$ where *e* is the natural base of the exponential function

(A) Introducing Base *e*

- So our original formula $\lim_{n\to\infty}\left(1000\times\left(1+\frac{0.1}{n}\right)^{(2\times n)}\right)$
- Now becomes A = $1000e^{0.1\times2}$ where the 0.1 was the interest rate, 2 was the length of the investment (2 years) and \$1000 was the original investment
- So our value becomes \$1221.40
- And our general equation can be written as A = Pe^{rt} where P is the original amount, r is the growth rate and t is the length of time
- Note that in this example, the growth happens continuously (i.e the idea that $n\to\infty$)

(B) Working with A = Pe^{rt}

- So our formula for situations featuring continuous change becomes A = Pe^{rt}
- In the formula, if r > 0, we have exponential growth and if r < 0, we have exponential decay
- P represents an initial amount

(C) Examples

- (i) I invest \$10,000 in a funding yielding 12% p.a. compounded continuously.
  - (a) Find the value of the investment after 5 years.
  - (b) How long does it take for the investment to triple in value?
- (ii) The population of the USA can be modeled by the eqn P(t) = $227e^{0.0093t}$, where P is population in millions and t is time in years since 1980
  - (a) What is the annual growth rate?
  - (b) What is the predicted population in 2015?
  - (c) What assumptions are being made in question (b)?
  - (d) When will the population reach 500 million?

(C) Examples

- (iii) A certain bacteria grows according to the formula $A(t) = 5000e^{0.4055t}$, where t is time in hours.
  - (a) What will the population be in 8 hours
  - (b) When will the population reach 1,000,000
- (iv) The function $P(t) = 1 - e^{-0.0479t}$ gives the percentage of the population that has seen a new TV show t weeks after it goes on the air.
  - (a) What percentage of people have seen the show after 24 weeks?
  - (b) Approximately, when will 90% of the people have seen the show?
  - (c) What happens to P(t) as t gets infinitely large? Why? Is this reasonable?

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(D) Working with e

- Use your calculator to determine the following:
- A) $e^{1.3}$
- B) $e^{0.25}$
- C) $e^{-1.5}$
- D) 1/e
- E) $e^{-1/3}$

- Graph $f(x) = e^x$ and on the same axis, graph $g(x) = 2^x$ and $h(x) = 3^x$
- Graph $g(x) = e^{-x}$ and on the same axis, graph $g(x) = 2^{-x}$ and $h(x) = 3^{-x}$

(E) The Natural Logarithm

- WE can use the base *e* in logarithms as well
- The expression $\log_e$ will now be called the **natural logarithm** and will be written as ln
- Therefore, ln(6) really means $\log_e(6)$ and will be read as what is the exponent on the base e that produces the power 6
- Using a calculator, ln 6 = 1.79176 meaning that $e^{1.79176} = 6$

(E) Working with the Natural Logarithm

- (A) Evaluate with your calculator:
- i) ln 50 ii) ln 100 iii) ln 2
- iv) ln0.56 v) ln (-0.5)
- (B) Evaluate without your calculator
- i) $\ln e^3$ ii) ln e iii) ln 1
- iv) $e^{\ln 2}$ v) $\ln e^{-2}$

(E) Working with the Natural Logarithm

- (C) Solve for x
- i) $e^x = 3$ ii) $e^{2x} = 7$
- iii) $e^{x-2} = 5$ iv) $e^{-x} = 6$
- v) $e^{2-x} = 13$ vi) $e^{3-x} = e2x$
- (D) Change the base from base 2 to base e
- i.e. Change from y = 2x to an equivalent $y = e^K$

(F) Homework

- Ex 5A, Q5,6,7,8,9,12
- Ex 5B #1, 5,6dg, 7abce
- from HM12, p103-104, Q3,5,6,9-13,16