T.2.7 - Solving Exponential & Logarithmic Equations

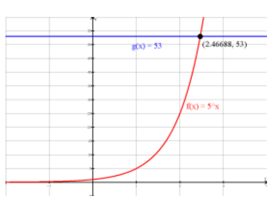
IB Math SL1 - Santowski

(A) Strategies for Solving Exponential Equations - Guessing

- we have explored a variety of equation solving strategies, namely being able to isolate a variable
- this becomes seemingly impossible for exponential equations like $5^x = 53$
- our earlier strategy was to express both sides of an equation with a common base, (i.e. $2^x = 32$) which we cannot do with the number 53 and the base of 5
- Alternatively, we can simply "guess & check" to find the right exponent on 5 that gives us 53 ➔ we know that $5^2 = 25$ and $5^3 = 125$, so the solution should be somewhere closer to 2 than 3

(B) Strategies for Solving Exponential Equations - Graphing

- Going back the example of $5^x = 53$, we always have the graphing option
- We simply graph $y_1 = 5^x$ and simultaneously graph $y_2 = 53$ and look for an intersection point (2.46688, 53)



(C) Strategies for Solving Exponential Equations - Inverses

- However, one general strategy that we have used previously was to use an "inverse" operation to isolate a variable
- and so now that we know how to "inverse" an exponential expression using logarithms, we will use the same strategy ➔ inverse an exponential using logarithms
- So then if $5^x = 53$, then $\log_5(53) = x$ ➔ but this puts us in the same dilemma as before ➔ we don't know the exponent on 5 that gives 53

(D) Strategies for Solving Exponential Equations - Logarithms

- So we will use the logarithm concept as we apply another logarithm rule ➔ let's simply take a common logarithm of each side of the equation ($\log_{10}$) (since our calculators are programmed to work in base 10)
- Thus, $5^x = 53$ now becomes
- $\log_{10}(5^x) = \log_{10}(53)$
- $\log_{10}(5)^x = \log_{10}(53)$
- $x[\log_{10}(5)] = \log_{10}(53)$ (using log rules)
- $x = \log_{10}(53) \div \log_{10}(5)$
- x = 2.46688 …..

(D) Strategies for Solving Exponential Equations – Natural Logarithms

- In solving $5^x = 53$, we used a common logarithm (log base 10) to solve the equation
- One other common logarithm you will see on your calculator is the natural logarithm (ln which uses a special base of numerical value 2.71828… which is notated by the letter *e* ➔ so loge(x) = ln(x))
- Thus, $\ln 5^x = \ln 53$
- And x(ln5) = ln53
- And x = ln53 ÷ ln5 = 2.46688 as before

(E) Examples

- Evaluate $\log_3 38 = x$
- Again, same basic problem ➔ we are using a base in which 38 is an awkward number to work with (unlike 9,27,81,243,729……)
- So let's change the expression to an exponential equation ➔ $3^x = 38$ and this puts us back to the point we were at before with $5^x = 53$!!
- Thus, $\log_{10}(3)^x = \log_{10}(38)$
- And xlog3 = log38
- So x = log38 ÷ log3 = 3.31107…..

(E) Examples

- Solve the following for x
- (a) $2^x = 8$ (b) $2^x = 16$
- (c) $2x = 11$ (d) $2^x = 12$
- (e) $2^{4x+1} = 8^{1-x}$ (e) $2^{x^2-4} = 8^x$
- (f) $2^{3x+2} = 9$ (g) $3(2^{2x-1}) = 4^{-x}$
- (h) $2^{4y+1} - 3^y = 0$

(E) Examples

- Solve the following for x and STATE restrictions for x (WHY). Verify your solutions:

- (a) $\log_2 x = 3$
- (b) $\log_2(x + 1) = 3$
- (c) ln10 – ln(7 – x) = lnx
- (d) 2logx – log4 = 3
- (e) 3 log(2*x* − 1) = 1
- (f) $\log_2 x + \log_2 7 = \log_2 21$
- (g) $\log_5(2x + 4) = 2$

(E) Examples

- (h) $2\log_9\sqrt{x} - \log_9(6x - 1) = 0$
- (i) log(x) + log(x – 1) = log(3x + 12)
- (j) $\log_2(x) + \log_2(x-2) = 3$
- (k) $\log_2(x^2 - 6x) = 3 + \log_2(1-x)$
- (l) log(x) = 1 – log(x – 3)
- (m) $\log_7(2x + 2) - \log_7(x - 1) = \log_7(x + 1)$

(F) Applications of Exponential Equations

- The half-life of radium-226 is 1620 years. After how many years is only 30 mg left if the original sample contained 150 mg?

- Recall the formula for half-life is $N(t) = N_0(2)^{(-t/h)}$ where h refers to the half-life of the substance

(F) Applications of Exponential Equations

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- Recall the formula for half-life is $N(t) = N_0(2)^{(-t/h)}$ where h refers to the half-life of the substance

- or $N(t) = N_0(1+r)^t$ where r is the rate of change (or common ratio of -0.5; and t would refer to the number of "conversion periods – or the number of halving periods)

- Therefore, $30 = 150(2)^{(-t/1620)}$
- $30/150 = 0.20 = 2^{(-t/1620)}$
- log(0.2) = (-t/1620) log (2)
- log (0.2) ÷ log (2) = -t/1620
- -1620 x log(0.2) ÷ log (2) = t
- Thus t = 3761.5 years

(F) Applications of Exponential Equations

- Two populations of bacteria are growing at different rates. Their populations at time *t* are given by $P_1(t) = 5^{t+2}$ and $P_2(t) = e^{2t}$ respectively. At what time are the populations the same?

(F) Applications of Exponential Equations

- The logarithmic function has applications for solving everyday situations:
- ex 1. Mr. S. drinks a cup of coffee at 9:45 am and his coffee contains 150 mg of caffeine. Since the half-life of caffeine for an average adult is 5.5 hours, determine how much caffeine is in Mr. S.'s body at class-time (1:10pm). Then determine how much time passes before I have 30 mg of caffeine in my body.
- ex 2. The value of the Canadian dollar , at a time of inflation, decreases by 10% each year. What is the half-life of the Canadian dollar?

(F) Applications of Exponential Equations

- ex 3. The half-life of radium-226 is 1620 a. Starting with a sample of 120 mg, after how many years is only 40 mg left?
- ex 4. Find the length of time required for an investment of $1000 to grow to $4,500 at a rate of 9% p.a. compounded quarterly.

(F) Applications of Exponential Equations

- ex 5. Dry cleaners use a cleaning fluid that is purified by evaporation and condensation after each cleaning cycle. Every time it is purified, 2% of the fluid is lost
- (a) An equipment manufacturer claims that after 20 cycles, about two-thirds of the fluid remains. Verify or reject this claim.
- (b) If the fluid has to be "topped up" when half the original amount remains, after how many cycles should the fluid be topped up?
- (c) A manufacturer has developed a new process such that two-thirds of the cleaning fluid remains after 40 cycles. What percentage of fluid is lost after each cycle?

(G) Internet Links

- College Algebra Tutorial on Exponential Equations (NOTE: this lesson uses natural logarithms to solve exponential equations)
- Solving Exponential Equations Lesson from Purple Math (NOTE: this lesson uses natural logarithms to solve exponential equations)
- SOLVING EXPONENTIAL EQUATIONS from SOS Math (NOTE: this lesson uses natural logarithms to solve exponential equations)

(H) Homework

- Nelson text, p132, Q2-12,14,16
- Math 12 P&P, p408,Q5-10
- IB Textbook: Ex 4D #1fg, 2bf;
- Ex 4E #4; Ex 5D #1bh; Ex 5E #1a,
- 3c