L56 – Discrete Random Variables, Distributions & Expected Values

IB Math SL1 - Santowski

Lesson Objectives

(A) Setting the Stage - Probabilities

- A bag contains 5 white marbles and 4 red marbles. Two marbles are selected, without replacement.
- (a) Present a tree diagram showing the possible outcomes
- (b) Determine the probability of selecting 0 white marbles
- (c) Determine the probability of selecting 1 white marble
- (d) Determine the probability of selecting 2 white marbles

(A) Setting the Stage - Probabilities

- Now, let's tabulate the probabilities from this experiment ➔ one row will be the calculated probabilities and the other row will be the number of white marbles selected

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Number of white marbles selected, x	0	1	2
Probability of selecting x white marbles	20/72	40/72	12/72

(A) Setting the Stage - Probabilities

- Now let's graph the data from our experiment
- So, now we can consider our probability data in the form of a table or graph and we will now refer to this data as a **probability distribution**
- We could also write equations to model the data in our tables or graphs **(probability distribution functions)**

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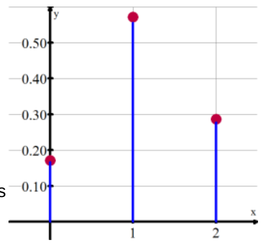

(B) Variables

- Recall the definition of a "variable" in stats ➔ the possible measureable outcomes in our data set/experiment
- Ex ➔ the number of students
- Ex ➔ the height of students
- Ex ➔ the volume of water consumed
- Ex ➔ the number of soda cans being recycled
- We have two types of variables that we consider in stats & probabilities ➔ continuous variables and discrete variables

(B) Variables

- **Continuous variables** would be variables (possible outcomes) such as student height, weight, student grades ➔ for a continuous variable, ANY value on an interval is possible
- **Discrete variables** would be variables (possible outcomes) such as number of students in classes, number of soda cans recycled, the number of races an athlete competed in

PRACTICE

- 29A, p710, Q1,2

(C) Discrete Random Variables

- Now back to our marbles experiment ➔ we tabulated the probability of the various outcomes in which we are interested
- All outcomes that we will now consider will be the number/count of the desired outcomes (number of white marbles) ➔ hence the idea of DISCRETE VARIABLES

(D) Notations

- Since we have introduced a new concept (probability distributions of discrete variables), we have some new notations to get used to
- We tend to use the letter X to represent the random variable we are measuring (the outcome)
- We use the letter x to represent the discrete numerical values that our variable, X, can have
- We use the notation $P(X = x) = p$ ➔ the probability that the variable X has a value of x

(D) Notations

- An example ➔ Consider the experiment of tossing a coin three times
- Our variable, X, will be (possible outcomes) the number of heads observed
- Our variable, X, will have certain discrete values that it can have ➔ x = 0,1,2,3
- So, the statement P(X = 2) would mean ➔ ???

PRACTICE

- A pair of dice are rolled. Let the variable X represent the sum of the numbers showing on the dice
- (a) Determine the possible values X can have
- (b) Display the probability distribution in a table
- (c) Display the probability distribution in a graph
- (d) Determine P(X = 8) and interpret

PRACTICE

- A fair coin is tossed 4 times. Let the variable X represent the number of heads that appear
- (a) Determine the number of possible values that X can have
- (b) Display this information on a table and a graph
- (c) Determine P(X > 1)
- (d) Determine P(X = 2)
- (e) Determine P(x < 3|X > 1)

(E) Laws of Probability Distributions

- (1) the probability of any one event occurring, p_i, is $0 \leq p_i \leq 1$
- (2) the sum of the probabilities of all possible outcomes is 1

$$\sum_{i=1}^{n} p_i = p_1 + p_2 + p_3 + \ldots + p_n = 1$$

PRACTICE

- The number of students that leave my class to go to the washroom can be modelled by the probability distribution function P(X = x) = *k*(3x + 1) where x = 0,1,2,3,4
- (a) Determine the value of k
- (b) Display this information on a table and a graph
- (c) Interpret P(X = 2) = 0.2
- (d) What are the chances that at least 2 students leave my room?

PRACTICE

- 29B, p 712, Q1,3,4,5

(F) Expected Values

- Example ➔ a single die ➔ You roll a die 240 times. How many 3's to you EXPECT to roll?
- (i.e. Determine the expectation of rolling a 3 if you roll a die 240 times)

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- (i.e. Determine the expectation of rolling a 3 if you roll a die 240 times)
- ANS ➔ 1/6 x 240 = 40 ➔ implies the formula of (n)x(p)
- BUT remember our focus now is not upon a single event (rolling a 3) but ALL possible outcomes and the resultant distribution of outcomes ➔ so .....

(F) Expected Values

- The **mean** of a random variable ➔ a measure of central tendency ➔ also known as its **expected value**,E(x), is **weighted average**. of all the values that a random variable would assume in the long run.

(F) Expected Value

- So back to the die ➔ what is the expected value when the die is rolled?
- Our weighted average is determined by sum of the products of outcomes and their probabilities $E(X) = \sum_i x_i \times p(x_i)$

(F) Expected Value

- Determine the expected value when rolling a six sided die

(F) Expected Value

- Determine the expected value when rolling a six sided die
- X = {1,2,3,4,5,6}
- $p(x_i) = 1/6$
- E(X) = (1)(1/6) + (2)(1/6) + (3)(1/6) + (4)(1/6) + (5)(1/6) + (6)(1/6)
- E(X) = 21/6 or 3.5

(F) Expected Value

- Ex. How many heads would you expect if you flipped a coin twice?

(F) Expected Value

- E(x) is not the value of the random variable x that you "expect" to observe if you perform the experiment **once**
- E(x) is a "**long run**" average; if you perform the experiment many times and observe the random variable x each time, then the average x of these observed x-values will get closer to E(x) as you observe more and more values of the random variable x.

(F) Expected Value

- Ex. How many heads would you expect if you flipped a coin twice?
- X = number of heads = {0,1,2}
- p(0)=1/4, p(1)=1/2, p(2)=1/4
- Weighted average = 0*1/4 + 1*1/2 + 2*1/4 = 1

(F) Expected Value

- A common application of expected value is gambling.
- For example, an American roulette wheel has 38 places where the ball may land, all equally likely.
- A winning bet on a single number pays 35-to-1, meaning that the original stake is not lost, and 35 times that amount is won, so you receive 36 times what you've bet.
- Considering all 38 possible outcomes, Determine the expected value of the profit resulting from a dollar bet on a single number

(F) Expected Value

- the expected value of the profit resulting from a dollar bet on a single number is the sum of potential net loss times the probability of losing and potential net gain times the probability of winning
- The net change in your financial holdings is −$1 when you lose, and $35 when you win, so your expected winnings are.....
- Outcomes are X = -$1 and X = +$35
- So E(X) = (-1)(37/38) + 35(1/38) = -0.0526
- Thus one may expect, on average, to lose about five cents for every dollar bet, and the **expected value** of a one-dollar bet is $0.9474.
- In gambling, an event of which the expected value equals the stake (i.e. the better's expected profit, or net gain, is zero) is called a "fair game".

(F) Expected Value

- Expectations can be used to describe the potential gains and losses from games.
- Ex. Roll a die. If the side that comes up is odd, you win the $ equivalent of that side. If it is even, you lose $4.
- Ex. Lottery – You pick 3 different numbers between 1 and 12. If you pick all the numbers correctly you win $100. What are your expected earnings if it costs $1 to play?

(F) Expected Value

- Ex. Roll a die. If the side that comes up is odd, you win the $ equivalent of that side. If it is even, you lose $4.
- Let X = your earnings

- X=1 P(X=1) = P({1}) =1/6
- X=3 P(X=1) = P({3}) =1/6
- X=5 P(X=1) = P({5}) =1/6
- X=-4 P(X=1) = P({2,4,6}) =3/6

- E(X) = 1*1/6 + 3*1/6 + 5*1/6 + (-4)*1/2
- E(X) = 1/6 + 3/6 +5/6 – 2= -1/2

(F) Expected Value

- Ex. Lottery – You pick 3 different numbers between 1 and 12. If you pick all the numbers correctly you win $100. What are your expected earnings if it costs $1 to play?

- Let X = your earnings
- X = 100-1 = 99
- X = -1

- P(X=99) = 1/(12 3) = 1/220
- P(X=-1) = 1-1/220 = 219/220
- E(X) = 100*1/220 + (-1)*219/220 = -119/220 = -0.54

(F) Expected Value

- The concept of Expected Value can be used to describe the expected monetary returns

- An investment in Project A will result in a ***loss*** of $26,000 with probability 0.30, break even with probability 0.50, or result in a profit of $68,000 with probability 0.20.
- An investment in Project B will result in a ***loss*** of $71,000 with probability 0.20, break even with probability 0.65, or result in a profit of $143,000 with probability 0.15.
- Which investment is better?

Tools to calculate E(X)-Project A

- Random Variable (X)- The amount of money received from the investment in Project A
- X can assume only x_1 , x_2 , x_3

- X= x_1 is the event that we have Loss
- X= x_2 is the event that we are breaking even
- X= x_3 is the event that we have a Profit

- x_1=$-26,000
- x_2=$0
- x_3=$68,000

- P(X= x_1)=0.3
- P(X= x_2)= 0.5
- P(X= x_3)= 0.2

Tools to calculate E(X)-Project B

- Random Variable (X)- The amount of money received from the investment in Project B
- X can assume only x_1 , x_2 , x_3

- X= x_1 is the event that we have Loss
- X= x_2 is the event that we are breaking even
- X= x_3 is the event that we have a Profit

- x_1=$-71,000
- x_2=$0
- x_3=$143,000

- P(X= x_1)=0.2
- P(X= x_2)= 0.65
- P(X= x_3)= 0.15

Tools to calculate E(X)-Project A & B

$$\begin{aligned}&\text{Project A}:\\&E(X) = 0.30\cdot(-\$26{,}000) + 0.50\cdot\$0 + 0.20\cdot\$68{,}000\\&\quad = \$5800\\&\text{Project B}:\\&E(X) = 0.20\cdot(-\$71{,}000) + 0.65\cdot\$0 + 0.15\cdot\$143{,}000\\&\quad = \$7250\end{aligned}$$