Lesson 62 - Binomial Expansion

IB Math SL1 - Santowski

Polynomial expansion

The binomial expansions

$(x+y)^0 =$

$(x+y)^1 =$

$(x+y)^2 =$

$(x+y)^3 =$

$(x+y)^4 =$

$(x+y)^5 =$

Polynomial expansion

The binomial expansions

$(x+y)^0 = 1$

$(x+y)^1 = x+y$

$(x+y)^2 = x^2+2xy+y^2$

$(x+y)^3 = x^3+3x^2y+3xy^2+y^3$

$(x+y)^4 = x^4+4x^3y+6x^2y^2+4xy^3+y^4$

$(x+y)^5 = x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$

reveal a pattern.

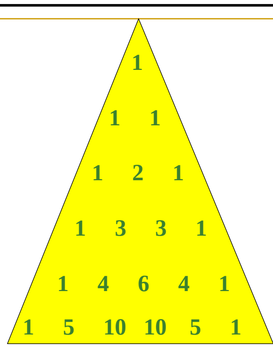
Polynomial expansion - A Binomial Expansion Pattern

- And the pattern is:
- This pattern is referred to as PASCAL'S TRIANGLE

	Row
1	0
1 1	1
1 2 1	2
1 3 3 1	3
1 4 6 4 1	4
1 5 10 10 5 1	5

- WHY is the pattern as it is???

This is good for lower powers but could get very large. We will introduce some notation to help us and **generalize** the coefficients with a formula based on what was observed here.



This is called Pascal's Triangle and would give us the coefficients for a binomial expansion of any power if we extended it far enough.

Applying Pascal's Triangle to Binomial Expansions

- Expand $(x + 2)^4$
- Expand $(2x – 3y)^4$
- Expand $\left(-3x^2 - \frac{2}{x}\right)^4$
- Find the leading coefficient of the x^{12} term in the expansion of $(2x – 3)^{21}$ ➔ in order to answer this question we need to know the WHY behind the pattern in the triangle.

Instead of x we have $2x$

Instead of a we have $-3y$

Let's use what we've learned to expand $(2x - 3y)^6$

Instead of x we have $2x$

Instead of a we have $-3y$

Let's use what we've learned to expand $(2x - 3y)^6$

First let's write out the expansion of the general $(x + a)^6$ and then we'll substitute.

$$(x+a)^6 = x^6 + \underline{6}\, ax^5 + \underline{15}\, a^2x^4 + \underline{20}\, a^3x^3 + \underline{15}\, a^4x^2 + \underline{6}\, a^5x + a^6$$

$$(2x-3y)^6 = (2x)^6 + 6(-3y)(2x)^5 + 15(-3y)^2(2x)^4 + 20(-3y)^3(2x)^3 + 15(-3y)^4(2x)^2 + 6(-3y)^5(2x) + (-3y)^6$$

$$= 64x^6 - 576x^5y + 2160x^4y^2 - 4320x^3y^3 + 4860x^2y^4 - 2916xy^5 + 729y^6$$

Polynomial expansion

- Consider $(x+y)^3$: $(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$
- Rephrase it as:

$$(x+y)(x+y)(x+y) = x^3 + [x^2y + x^2y + x^2y] + [xy^2 + xy^2 + xy^2] + y^3$$

- When **choosing** x twice and y once, there are 3 ways to choose where the x comes from
- When **choosing** x once and y twice, there are 3 ways to choose where the y comes from

Combinations

There are 5 top students in this class. If I would like to select 2 students out of these five to represent this class. How many ways are there for my choice?

List of the combinations (order is not considered) :

(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5), (3,4), (3,5), (4,5)

A symbol is introduced to represent this selection.

$_nC_r$ or ^nC_r or C^n_r or C(n,r)

! The Factorial Symbol !

$0! = 1 \quad 1! = 1$

$n! = n(n-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1$

n must be an integer greater than or equal to 2

What this says is if you have a positive integer followed by the factorial symbol you multiply the integer by each integer less than it until you get down to 1.

$$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$$

! Your calculator can compute factorials. The ! symbol is under the "math" menu and then "prob". !

If r and n are integers with $0 \le r \le n$, the symbol $\binom{n}{r}$ is defined as

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

This symbol is read "n taken r at a time"

Your calculator can compute these as well. It is also under the "math" and then "prob" menu and is usually denoted *nCr* with the *C* meaning combinations. In probability, there are n things to choose from and you are choosing j of them for various combinations.

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Let's work a couple of these:

$$\binom{5}{2} = \frac{5!}{2!(5-2)!} = \frac{5\cdot 4\cdot 3\cdot 2\cdot 1}{2\cdot 1(3\cdot 2\cdot 1)} = \frac{20}{2} = 10$$

$$\binom{12}{9} = \frac{12!}{9!(12-9)!} = \frac{\overset{2}{12}\cdot 11\cdot 10\cdot 9!}{9!(3\cdot 2\cdot 1)} = 220$$

We are now ready to see how this applies to expanding binomials.

Polynomial expansion

- Consider $(x+y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5$
- To obtain the x^5 term
  - Each time you multiple by $(x+y)$, you select the x
  - Thus, of the 5 choices, you choose x 5 times
    - C(5,5) = 1
  - Alternatively, you choose y 0 times
    - C(5,0) = 1
- To obtain the x^4y term
  - Four of the times you multiply by $(x+y)$, you select the x
    - The other time you select the y
  - Thus, of the 5 choices, you choose x 4 times
    - C(5,4) = 5
  - Alternatively, you choose y 1 time
    - C(5,1) = 5
- To obtain the x^3y^2 term
  - C(5,3) = C(5,2) = 10

The Binomial Theorem (Binomial expansion)

$$(a+b)^5 = 1a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + 1b^4$$

$$(a+b)^5 = 1a^5 + {}_5C_4a^4b + {}_5C_3a^3b^2 + {}_5C_2a^2b^3 + {}_5C_1ab^4 + {}_5C_0b^4$$

$$(a+b)^n = 1a^n + {}_nC_{n-1}a^{n-1}b + {}_nC_{n-2}a^{n-2}b^2 + {}_nC_{n-3}a^{n-3}b^3 + \ldots + {}_nC_{n-r}a^{n-r}b^r + \ldots + 1b^n$$

where n is a positive integer

Polynomial expansion: The binomial theorem

- For $(x+y)^n$

$$(x+y)^n = \binom{n}{n}x^ny^0 + \binom{n}{n-1}x^{n-1}y^1 + \cdots + \binom{n}{1}x^1y^{n-1} + \binom{n}{0}x^0y^n$$

- OR

$$(x+y)^n = \binom{n}{0}x^ny^0 + \binom{n}{1}x^{n-1}y^1 + \cdots + \binom{n}{n-1}x^1y^{n-1} + \binom{n}{n}x^0y^n$$

Polynomial expansion: Binomial Coefficients

Binomial Coefficient

For nonnegative integers n and r, with $r \le n$,

$$_nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Example

- Find the coefficient of x^5y^8 in $(x+y)^{13}$
- Answer:

Example

- Find the coefficient of x^5y^8 in $(x+y)^{13}$
- Answer: $\binom{13}{5}=\binom{13}{8}=1287$

Example

- Find the 5th term of the expansion of $(x + a)^{12}$
- Completely expand $(x + a)^{12}$

Here is the expansion of $(x + a)^{12}$

$$x^{12} + 12\,a\,x^{11} + 66\,a^2\,x^{10} + 220\,a^3\,x^9 + 495\,a^4\,x^8 + 792\,a^5\,x^7 + 924\,a^6\,x^6 + 792\,a^7\,x^5 + 495\,a^8\,x^4 + 220\,a^9\,x^3 + 66\,a^{10}\,x^2 + 12\,a^{11}\,x + a^{12}$$

…and the 5th term matches the term we obtained!

In this expansion, observe the following:

•Powers on *a* and *x* add up to power on binomial

•*a's* increase in power as *x*'s decrease in power from term to term.

•Powers on *a* are one less than the term number

•Symmetry of coefficients (i.e. 2nd term and 2nd to last term have same coefficients, 3rd & 3rd to last etc.) so once you've reached the middle, you can copy by symmetry rather than compute coefficients.

Examples

- What is the coefficient of $x^{12}y^{13}$ in $(x+y)^{25}$?
- What is the coefficient of $x^{12}y^{13}$ in $(2x-3y)^{25}$?
  - Rephrase it as $(2x+(-3y))^{25}$
  - The coefficient occurs when j=13:

Examples

- What is the coefficient of $x^{12}y^{13}$ in $(x+y)^{25}$?

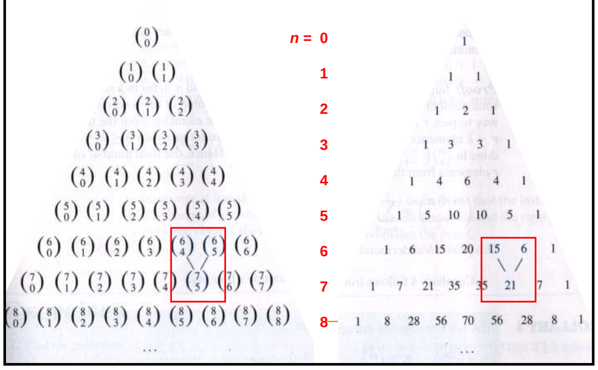
  $\binom{25}{13}=\binom{25}{12}=\frac{25!}{13!12!}=5{,}200{,}300$
- What is the coefficient of $x^{12}y^{13}$ in $(2x-3y)^{25}$?
  - Rephrase it as $(2x+(-3y))^{25}$

    $(x+y)^{25}=(2x+(-3y))^{25} \Rightarrow$

    use $(2x)$ instead of x and $(-3y)$ instead of y
  - The coefficient occurs when r =12:

$$\binom{25}{12}(2x)^{12}(-3y)^{13}=\frac{25!}{13!12!}2^{12}(-3)^{13}x^{12}y^{13}=-33{,}959{,}763{,}545{,}702{,}400$$

Pascal's triangle Revisited



Example

- Determine the coefficient of the x^7 term in the expansion of $\left(x^2 + \frac{4}{x}\right)^{11}$

Example

- Determine the coefficient of the x^7 term in the expansion of

- $Term = \binom{11}{r}(x^2)^r\left(\frac{4}{x}\right)^{11-r}$

  $Term = \binom{11}{r}(x^{2r})\left(\frac{4^{11-r}}{x^{11-r}}\right)$

  $Term = \binom{11}{r}\left(\frac{x^{2r}}{x^{11-r}}\right)(4^{11-r})$

  $Term = \binom{11}{r}(x^{3r-11})(4^{11-r})$

  so $x^7 = x^{3r-11}$

  so $7 = 3r - 11 \Rightarrow r = 6$

  $\therefore term = \binom{11}{6}(4^5)x^7 = 473088x^7$

Example

Extension to Trinomial

Expand $(1 - x + x^2)^4$

Example

Extension to Trinomial

$(1 - x + x^2)^4$

$= [1 - x(1 - x)]^4$

$= 1^4 - {}_4C_3(1)^3x(1-x) + {}_4C_2(1)^2x^2(1-x)^2$

$\quad - {}_4C_1(1)^1x^3(1-x)^3 + x^4(1-x)^4$

Homework

- HW
- - Ex 9A #1bdh, 2cf, 3b, 4abii, 8a
- - Ex 9B #1ab, 2ad, 3ac, 4b;

Quick survey

- I felt I understood the material in this slide set…

a) Very well
b) With some review, I'll be good
c) Not really
d) Not at all

Quick survey

- The pace of the lecture for this slide set was…

a) Fast
b) About right
c) A little slow
d) Too slow

Quick survey

- How interesting was the material in this slide set? Be honest!

a) Wow! That was SOOOOOO cool!
b) Somewhat interesting
c) Rather borting
d) Zzzzzzzzzzz