Binomial Distributions

IB Math SL1 - Santowski

The Binomial Setting

- Fixed number of *n* trials
- Independence
- Two possible outcomes: success or failure
- Same probability of a success for each observation
- If it FITS, it's binomial.

Binomial example

- The Binomial Distribution is the distribution of COUNTS.
- It counts the number of successes in a certain number of trials.
- Take the example of 5 coin tosses. What's the probability that you flip exactly 3 heads in 5 coin tosses?

Binomial distribution

Solution:
One way to get exactly 3 heads: HHHTT

What's the probability of this exact arrangement?
P(heads)xP(heads) xP(heads)xP(tails)xP(tails)
$=(1/2)^3 x (1/2)^2$

Another way to get exactly 3 heads: THHHT
Probability of this exact outcome = $(1/2)^1 x (1/2)^3 x (1/2)^1 = (1/2)^3 x (1/2)^2$

Binomial distribution

In fact, $(1/2)^3 x (1/2)^2$ is the probability of each unique outcome that has exactly 3 heads and 2 tails.

So, the overall probability of 3 heads and 2 tails is:
$(1/2)^3 x (1/2)^2 + (1/2)^3 x (1/2)^2 + (1/2)^3 x (1/2)^2 + \ldots$.. for as many unique arrangements as there are—but how many are there??

$\binom{5}{3}$ ways to arrange 3 heads in 5 trials

Outcome	Probability
THHHT	$(1/2)^3 x (1/2)^2$
HHHTT	$(1/2)^3 x (1/2)^2$
TTHHH	$(1/2)^3 x (1/2)^2$
HTTHH	$(1/2)^3 x (1/2)^2$
HHTTH	$(1/2)^3 x (1/2)^2$
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10 arrangements $x (1/2)^3 x (1/2)^2$

The probability of each unique outcome (note: they are all equal)

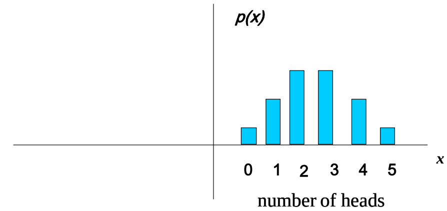
$_5C_3 = 5!/3!2! = 10$

$\therefore$ **P(3 heads and 2 tails) =** $\binom{5}{3}$ *x P(heads)*^{3} *x P(tails)*^{2} *=*

10 x (½)^{5}=31.25%

Binomial distribution function:

X= the number of heads tossed in 5 coin tosses



Example 2

As voters exit the polls on Nov. 4, you ask a representative random sample of 6 voters if they voted for Obama. If the true percentage of voters who vote for Obama on Nov. 4 is 55.1%, what is the probability that, *in your sample,* exactly 2 voted for Obama and 4 did not?

Solution:

$\binom{6}{2}$ ways to arrange 2 Obama votes among 6 voters

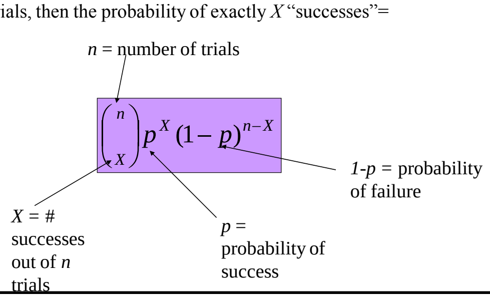
Outcome		Probability
OONNNN		$= (.551)^2 x (.449)^4$
NOONNN	$(.449)^1 x (.551)^2 x (.449)^3$	$= (.551)^2 x (.449)^4$
NNOONN	$(.449)^2 x (.551)^2 x (.449)^2$	$= (.551)^2 x (.449)^4$
NNNOON	$(.449)^3 x (.551)^2 x (.449)^1$	$= (.551)^2 x (.449)^4$
NNNNOO	$(.449)^4 x (.551)^2$	$= (.551)^2 x (.449)^4$
.		
.		

15 arrangements $x (.551)^2 x (.449)^4$

$\therefore$P(2 yes votes exactly) = $\binom{6}{2} x (.551)^2 x (.449)^4 = 18.5\%$

Binomial distribution, generally

Note the general pattern emerging → if you have only two possible outcomes (call them 1/0 or yes/no or success/failure) in *n* independent trials, then the probability of exactly *X* "successes"=



Definitions: Binomial

- **Binomial:** Suppose that *n* independent experiments, or trials, are performed, where *n* is a fixed number, and that each experiment results in a "success" with probability *p* and a "failure" with probability *1-p*. The total number of successes, *X*, is a binomial random variable with parameters *n* and *p*.
- We write: ***X ~ Bin (n, p)*** *{reads: "X is distributed binomially with parameters n and p}*
- And the probability that *X=r* (i.e., that there are <u>exactly</u> *r* successes) is: $P(X=r) = \binom{n}{r} p^r (1-p)^{n-r}$

What's the difference between binompdf and binomcdf?

- If we were looking for the probability of getting 6 heads out of 10 tosses, then binompdf only finds the likelihood of getting 6 successes.
- Binomcdf adds up all the probability of successes up to that certain number, 6 in this case, of successes, starting from 0 to k.

Binomial distribution: example

- If I toss a coin 20 times, what's the probability of getting exactly 10 heads?

Binomial distribution: example

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$$\binom{20}{10}(.5)^{10}(.5)^{10} = .176$$

Binomial distribution: example

- If I toss a coin 20 times, what's the probability of getting of getting 2 or fewer heads?

Binomial distribution: example

- If I toss a coin 20 times, what's the probability of getting of getting 2 or fewer heads?

$$\binom{20}{0}(.5)^{0}(.5)^{20} = \frac{20!}{20!0!}(.5)^{20} = 9.5x10^{-7} +$$
$$\binom{20}{1}(.5)^{1}(.5)^{19} = \frac{20!}{19!1!}(.5)^{20} = 20x9.5x10^{-7} = 1.9x10^{-5} +$$
$$\binom{20}{2}(.5)^{2}(.5)^{18} = \frac{20!}{18!2!}(.5)^{20} = 190x9.5x10^{-7} = 1.8x10^{-4}$$
$$= 1.8x10^{-4}$$

**All probability distributions are characterized by an expected value and a variance:

If *X* follows a binomial distribution with parameters *n* and *p*: ***X ~ Bin (n, p)***

Then:

$\mu_x = E(X) = np$

$\sigma_x^2 = Var(X) = np(1-p)$

$\sigma_x^2 = SD(X) = \sqrt{np(1-p)}$

Note: the variance will always lie between 0*N-.25 *N

p(1-p) reaches maximum at p=.5

P(1-p)=.25

Practice problems

- You are performing a study. If the probability of developing disease in the exposed group is .05 for the study duration, then if you sample (randomly) 500 exposed people, what's the probability that **at most** 10 exposed people develop the disease?

Answer

What's the probability that **at most** 10 exposed subjects develop the disease?

This is asking for a CUMULATIVE PROBABILITY: the probability of 0 getting the disease or 1 or 2 or 3 or 4 or up to 10.

$P(X \leq 10) = P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + \ldots + P(X=10) =$

$$\binom{500}{0}(.05)^0(.95)^{500} + \binom{500}{1}(.05)^1(.95)^{499} + \binom{500}{2}(.05)^2(.95)^{498} + \ldots + \binom{500}{10}(.05)^{10}(.95)^{490} < .01$$

Exploring "Shortcuts"

- Take the example of 5 tosses of a coin. What's the probability that you get 3 heads in the 5 tosses?
- Or getting at least 2 heads?

A brief distraction: Pascal's Triangle Trick

You'll rarely calculate the binomial by hand. However, it is good to know how to …

Pascal's Triangle Trick for calculating binomial coefficients

Recall from math in your past that Pascal's Triangle is used to get the coefficients for binomial expansion…

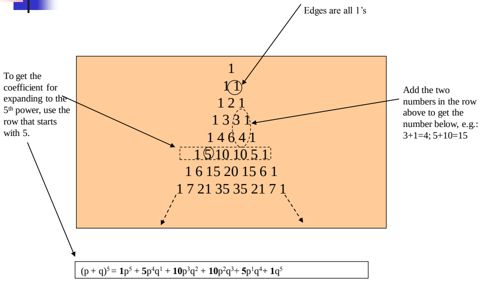
For example, to expand: $(p + q)^5$

The powers follow a set pattern: $p^5 + p^4q^1 + p^3q^2 + p^2q^3 + p^1q^4 + q^5$

But what are the coefficients?

- Use Pascal's Magic Triangle…

Pascal's Triangle



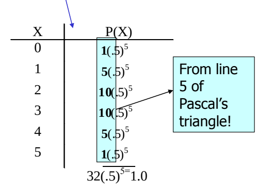
Same coefficients for X~Bin(5,p)

For example, X=# heads in 5 coin tosses:

X	P(X)
0	$\binom{5}{0}(.5)^0(.5)^5$
1	$\binom{5}{1}(.5)^1(.5)^4$
2	$\binom{5}{2}(.5)^2(.5)^3$
3	$\binom{5}{3}(.5)^3(.5)^2$
4	$\binom{5}{4}(.5)^4(.5)^1$
5	$\binom{5}{5}(.5)^5(.5)^0$

$\binom{5}{0}$=5!/0!5!=1 $\binom{5}{1}$=5!/1!4! = 5 $\binom{5}{2}$ = 5!/2!3!=5x4/2=10 $\binom{5}{3}$=5!/3!2!=10

$\binom{5}{4}$=5!/4!1!= 5 $\binom{5}{5}$=5!/5!1!=1 (Note the symmetry!)



Relationship between binomial probability distribution and binomial expansion

If p + q = 1 (which is the case if they are binomial probabilities)

then: $(p + q)^5 = (1)^5 = 1$ or, equivalently:

$1p^5 + 5p^4q^1 + 10p^3q^2 + 10p^2q^3 + 5p^1q^4 + 1q^5 = 1$

(the probabilities sum to 1, making it a probability distribution!)

P(X=0) P(X=1) P(X=2) P(X=3) P(X=4) P(X=5)

Practice problems

If the probability of being a smoker among a group of cases with lung cancer is .6, what's the probability that in a group of 8 cases you have:

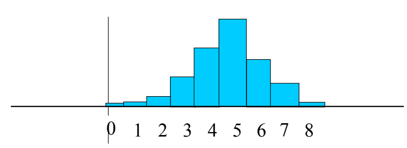
(a) Three smokers
(b) less than 2 smokers?
(c) More than 5?

Answer

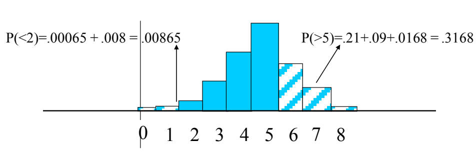
X	P(X)
0	$1(.4)^8$=.00065
1	$8(.6)^1(.4)^7$=.008
2	$28(.6)^2(.4)^6$=.04
3	$56(.6)^3(.4)^5$=.12
4	$70(.6)^4(.4)^4$=.23
5	$56(.6)^5(.4)^3$=.28
6	$28(.6)^6(.4)^2$=.21
7	$8(.6)^7(.4)^1$=.090
8	$1(.6)^8$=.0168

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1

Answer, continued



Answer, continued



E(X) = 8 (.6) = 4.8
Var(X) = 8 (.6) (.4) =1.92
StdDev(X) = 1.38

Examples

- Suppose a die is tossed 5 times. What is the probability of getting exactly 2 fours?

Examples - Solution

- Suppose a die is tossed 5 times. What is the probability of getting exactly 2 fours?
- *Solution:* This is a binomial experiment in which the number of trials is equal to 5, the number of successes is equal to 2, and the probability of success on a single trial is 1/6 or about 0.167. Therefore, the binomial probability is:
- binompdf(5, 0.167,2) = $_5C_2 * (1/6)^2 * (5/6)^3$
- binompdf(5, 0.167,2) = 0.161

Examples

- A **cumulative binomial probability** refers to the probability that the binomial random variable falls within a specified range (e.g., is greater than or equal to a stated lower limit and less than or equal to a stated upper limit).
- For example, we might be interested in the cumulative binomial probability of obtaining 45 or fewer heads in 100 tosses of a coin

Examples - Solution

- For example, we might be interested in the cumulative binomial probability of obtaining 45 or fewer heads in 100 tosses of a coin
- This would be the sum of all these individual binomial probabilities.
- b(x $\leq$ 45; 100, 0.5) = b(x = 0; 100, 0.5) + b(x = 1; 100, 0.5) + ... + b(x = 44; 100, 0.5) + b(x = 45; 100, 0.5)

Examples

- The probability that a student is accepted to a prestigeous college is 0.3. If 5 students from the same school apply, what is the probability that at most 2 are accepted?

Examples - Solution

- The probability that a student is accepted to a prestigeous college is 0.3. If 5 students from the same school apply, what is the probability that at most 2 are accepted?
- *Solution:* To solve this problem, we compute 3 individual probabilities, using the binomial formula. The sum of all these probabilities is the answer we seek. Thus,
- b(x $\leq$ 2; 5, 0.3) = b(x = 0; 5, 0.3) + b(x = 1; 5, 0.3) + b(x = 2; 5, 0.3)
- b(x $\leq$ 2; 5, 0.3) = 0.1681 + 0.3601 + 0.3087
- b(x $\leq$ 2; 5, 0.3) = 0.8369

Examples

- If a student randomly guesses at five multiple-choice questions, find the probability that the student gets exactly three correct. Each question has five possible choices.
-

Examples - Solutions

- If a student randomly guesses at five multiple-choice questions, find the probability that the student gets exactly three correct. Each question has five possible choices.
- Solution
- In this case n = 5, X = 3, and p = 1/5, since there is one chance in five of guessing a correct answer. Then,
- $$P(3)=\frac{5!}{(5-3)!3!}\cdot\left(\frac{1}{5}\right)^3\left(\frac{4}{5}\right)^2=0.05$$

Examples

- A survey from Teenage Research Unlimited (Northbrook, Ill.) found that 30% of teenage consumers receive their spending money from part-time jobs. If five teenagers are selected at random, find the probability that at least three of them will have part-time jobs.

Examples - Solutions

- A survey from Teenage Research Unlimited (Northbrook, Ill.) found that 30% of teenage consumers receive their spending money from part-time jobs. If five teenagers are selected at random, find the probability that at least three of them will have part-time jobs.
- To find the probability that at least three have a part-time job, it is necessary to find the individual probabilities for either 3, 4, or 5 and then add them to get the total probability.

$$P(3)=\frac{5!}{(5-3)!3!}\cdot(0.3)^3(0.7)^2=0.132$$

$$P(4)=\frac{5!}{(5-4)!4!}\cdot(0.3)^4(0.7)^1=0.028$$

$$P(5)=\frac{5!}{(5-5)!5!}\cdot(0.3)^5(0.7)^0=0.002$$

- Hence, P(at least three teenagers have part-time jobs) = 0.132 + 0.028 + 0.002 = 0.162

Homework

- Ex 29E, Q1-9
- More Examples