4.3 Zeros of Quadratic Functions

You have seen that a quadratic function, $f(x)$, may have no zeros, one zero, or two zeros. The zeros, or x-intercepts, occur when $f(x) = 0$. There are several ways to find the number of zeros for a quadratic function.

Part 1: Using a Graph to Find the Number of Zeros

Find the number of zeros in the graph of a quadratic function by finding the number of times the graph crosses or touches the x-axis.

Example 1

The demand function for a new mechanical part is $p(x) = -0.5x + 7.8$, where p is the price in dollars and x is the quantity sold in thousands. The new part can be manufactured by three different processes, A, B, or C. The cost function for each process is as follows:

Process A: $C(x) = 4.6x + 5.12$
Process B: $C(x) = 3.8x + 5.12$
Process C: $C(x) = 5.3x + 3.8$

Use a graphing calculator to investigate the break-even quantities for each process. Which process would you recommend to the company?

Solution

The profit function for each process is as follows:

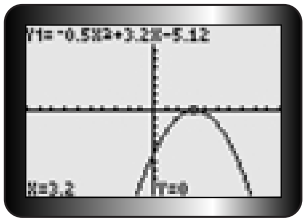
Process A: $$P_A(x) = x(-0.5x + 7.8) - (4.6x + 5.12) = -0.5x^2 + 3.2x - 5.12$$

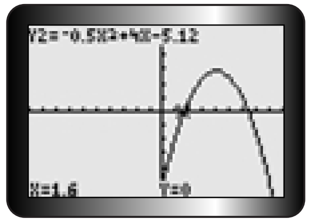
Process B: $$P_B(x) = x(-0.5x + 7.8) - (3.8x + 5.12) = -0.5x^2 + 4x - 5.12$$

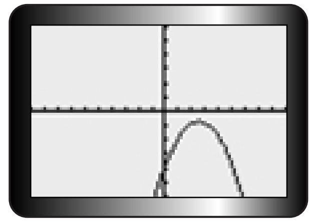
Process C: $$P_C(x) = x(-0.5x + 7.8) - (5.3x + 3.8) = -0.5x^2 + 2.5x - 3.8$$

The company will break even when the profit is 0, that is, $P(x) = 0$.

Here are the graphs of these profit functions:







For process A, the corresponding graph of the profit function has only one x-intercept. If $x = 3.2$, then $P_A(x) = 0$. If 3200 parts were sold, then the company would not make a profit, but would break even.

For process B, the corresponding graph of the profit function has two x-intercepts. If $x = 1.6$, or $x = 6.4$, then $P_B(x) = 0$. If 1600 parts are sold, then the company breaks even. The profit increases to a maximum if 4000 parts are sold. Then the profit tapers to 0 if 6400 parts are sold.

For process C, the corresponding graph of the profit function has no x-intercepts. $P_C(x) \neq 0$. The company would not make a profit; instead, it would lose money.

The company would profit most by using process B.

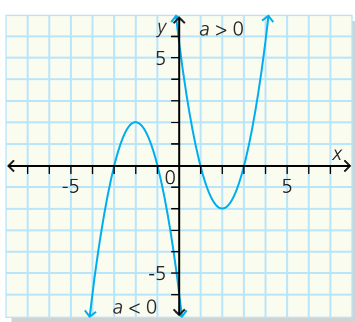
Part 2: Finding the Number of Zeros Algebraically

The three forms of a quadratic function are the **factored form**, $f(x) = a(x - p)(x - q)$, the **vertex form**, $f(x) = a(x - h)^2 + k$, and the **standard form**, $f(x) = ax^2 + bx + c$. You can find the number of zeros for a function that is in any of these forms.

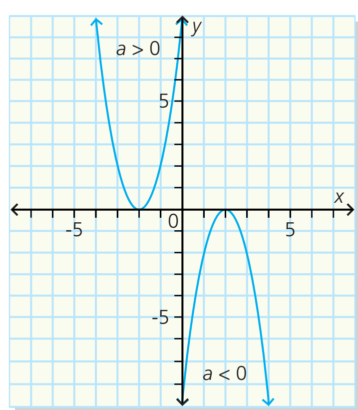
Finding the Number of Zeros For a Quadratic Function in Factored Form

The zeros are clearly visible in any quadratic function in factored form. A zero occurs on the graph of the relation whenever any of the factors in the relation equals 0.

The quadratic function $f(x) = a(x - p)(x - q)$ has two zeros at $x = p$ and $x = q$.



The quadratic function $g(x) = a(x - p)^2$ has one zero at $x = p$.



Example 2

Without drawing the graph, find the number of zeros for each function.

(a) $f(x) = -0.4(x - 1)(x + 7)$ **(b)** $g(x) = 3(x - 2.4)^2$

Solution

(a) Since $f(x) = -0.4(x - 1)(x + 7)$ is in factored form, the zeros are visible.

If $f(x) = 0$, then $x - 1 = 0$ or $x + 7 = 0$

$x = 1$ $\quad$ $x = -7$

The function $f(x)$ has two zeros.

(b) The zero is also visible in $g(x) = 3(x - 2.4)^2$.

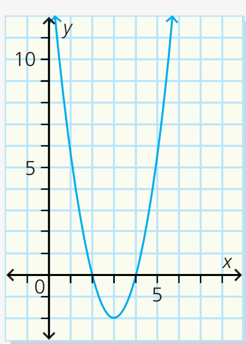
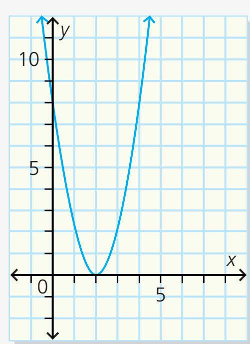
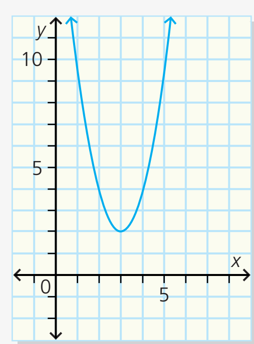
If $g(x) = 0$, then $x - 2.4 = 0$

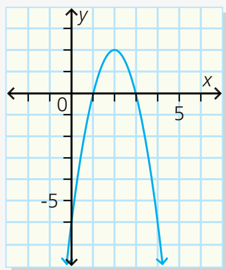
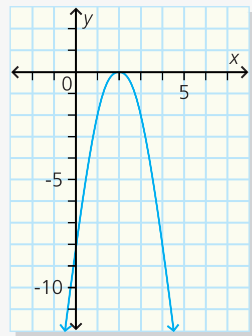
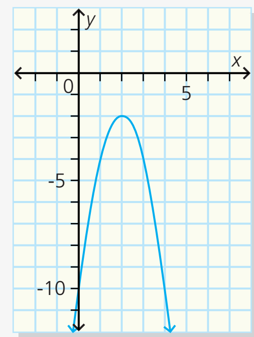
$x = 2.4$

The function $g(x)$ has one zero.

Finding the Number of Zeros for a Quadratic Function in Vertex Form

A quadratic function in vertex form shows the direction of the parabola's opening and the location of the vertex in terms of the x-axis.

For $f(x) = a(x - h)^2 + k$, $\boldsymbol{a > 0}$ the parabola opens **up**. The value of k determines the number of zeros.		
If $k < 0$, then $f(x)$ has two zeros. The vertex (h, k) lies below the x-axis.	If $k = 0$, then $f(x)$ has one zero. The vertex (h, k) lies on the x-axis.	If $k > 0$, then $f(x)$ has no zeros. The vertex (h, k) lies above the x-axis.
		

<table>
<tr><th colspan="3">For $f(x) = a(x - h)^2 + k$, $a < 0$ the parabola opens down.
The value of k determines the number of zeros.</th></tr>
<tr>
<td>If $k > 0$, then $f(x)$ has two zeros. The vertex (h, k) lies above the x-axis.

</td>
<td>If $k = 0$, then $f(x)$) has one zero. The vertex (h, k) lies on the x-axis.

</td>
<td>If $k < 0$, then $f(x)$ has no zeros. The vertex (h, k) lies below the x-axis.

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A quadratic function in vertex form, $f(x) = a(x - h)^2 + k$, has

- two zeros if a and k have opposite signs
- one zero if $k = 0$
- no zeros if a and k have the same signs

Example 3

Without drawing the graph, find the number of zeros for each function.

(a) $g(x) = 1.3(x - 4)^2 + 2.2$

(b) $h(x) = -1.7(x + 2)^2 + 4.5$

(c) $f(x) = 3(x - 2.4)^2$

Solution

(a) Since $g(x) = 1.3(x - 4)^2 + 2.2$ is in vertex form with $a > 0$ and $k > 0$, the parabola opens up and the vertex (4, 2.2) lies above the x-axis. $g(x)$ has no zeros. Alternatively, since a and k have the same signs, $g(x)$ has no zeros.

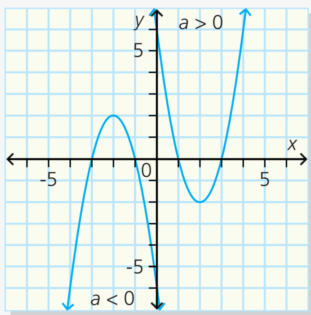
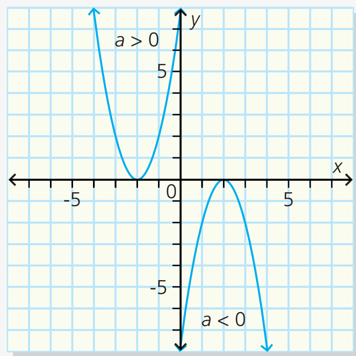
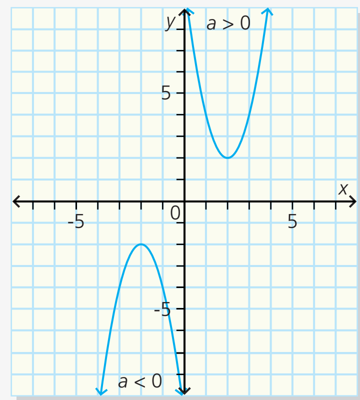
(b) Since $h(x) = -1.7(x + 2)^2 + 4.5$ is in vertex form with $a < 0$ and $k > 0$, the parabola opens down and the vertex $(-2, 4.5)$ lies above the x-axis. $h(x)$ has two zeros. Alternatively, since a and k have opposite signs, $h(x)$ has two zeros.

(c) Since $f(x) = 3(x - 2.4)^2$ is in vertex form with $a > 0$ and $k = 0$, the parabola opens up and the vertex (2.4, 0) lies on the x-axis. $f(x)$ has one zero. Alternatively, since $k = 0$, $f(x)$ has one zero.

Finding the Number of Zeros for a Quadratic Function in Standard Form

To find the zeros of the function $f(x) = ax^2 + bx + c$, you would solve $f(x) = 0$ or the equation $ax^2 + bx + c = 0$, which has roots $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The expression $b^2 - 4ac$ in the quadratic formula is called the **discriminant**. The discriminant tells the number of zeros for a quadratic function.

<table>
<tr><th>If $b^2 - 4ac > 0$, the quadratic function has two zeros.</th><th>If $b^2 - 4ac = 0$, the quadratic function has one zero.</th><th>If $b^2 - 4ac < 0$, the quadratic function has no zeros.</th></tr>
<tr><td></td><td></td><td></td></tr>
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Example 4

Without drawing the graph, determine the number of zeros for each function.

(a) $h(x) = -4.9x^2 + 24.5x - 30.625$ **(b)** $g(x) = 3.2x^2 + 11.4x + 26.6$

(c) $f(x) = 2.1x^2 + 3.5x - 1.2$

Solution

Each function is in standard form. Calculate the value of $b^2 - 4ac$.

(a)
$$\begin{aligned} b^2 - 4ac &= (24.5)^2 - 4(-4.9)(-30.625) \\ &= 0 \end{aligned}$$

Since $b^2 - 4ac = 0$, the function $h(x)$ has one zero.

(b)
$$\begin{aligned} b^2 - 4ac &= (11.4)^2 - 4(3.2)(26.6) \\ &= -210.52 \end{aligned}$$

Since $b^2 - 4ac < 0$, the function $g(x)$ has no zeros.

(c)
$$\begin{aligned} b^2 - 4ac &= (3.5)^2 - 4(2.1)(-1.2) \\ &= 22.33 \end{aligned}$$

Since $b^2 - 4ac > 0$, the function $f(x)$ has two zeros.

The Number of Roots of a Quadratic Equation

A quadratic equation in the form $ax^2 + bx + c = 0$ has real roots that correspond to the zeros of the function $f(x) = ax^2 + bx + c$. A quadratic equation may have no real roots, one real root, or two distinct real roots.

Example 5

For what value(s) of k does the equation $kx - 8 = 2x^2$ have

(a) two distinct real roots? **(b)** one real root? **(c)** no real roots?

Solution

(a) For two distinct real roots, $b^2 - 4ac > 0$.

$kx - 8 = 2x^2$	Rearrange the equation.
$-2x^2 + kx - 8 = 0$	Substitute a, b, and c into the discriminant.
$k^2 - 4(-2)(-8) > 0$	
$k^2 > 64$	Take the square roots.
$k < -8$ or $k > 8$	

(b) For one real root, $b^2 - 4ac = 0$. Use the equation $-2x^2 + kx - 8 = 0$.

$-2x^2 + kx - 8 = 0$	Substitute a, b, and c into the discriminant.
$k^2 - 4(-2)(-8) = 0$	
$k^2 = 64$	Take the square roots.
$k = -8$ or $k = 8$	

(c) For no real roots, $b^2 - 4ac < 0$. Use the equation $-2x^2 + kx - 8 = 0$.

$-2x^2 + kx - 8 = 0$	Substitute a, b, and c into the discriminant.
$k^2 - 4(-2)(-8) < 0$	
$k^2 < 64$	Take the square roots.
$k > -8$ and $k < 8$	

$\therefore -8 < k < 8$

Consolidate Your Understanding

1. What are the zeros of a function? How are the zeros related to the x-intercepts of the graph of the function?

2. How many zeros does a quadratic function have? How can you determine the number of zeros

(a) from the graph? **(b)** from the factored form?

(c) from the vertex form? **(d)** from the standard form?

3. (a) Draw a sketch that shows all the ways that $f(x) = ax^2 + bx + c$ can intersect the x-axis.

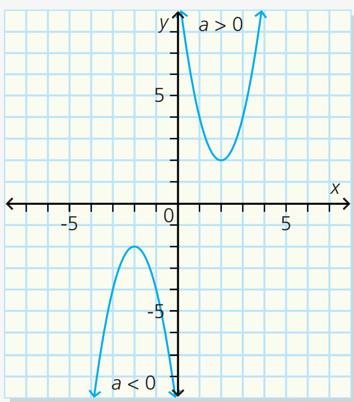
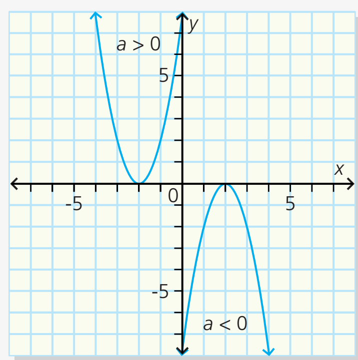
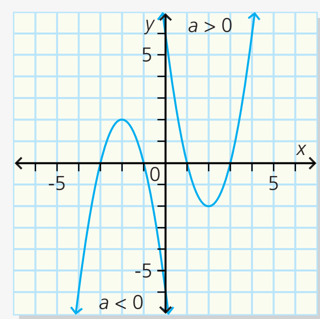
(b) Draw a sketch that shows all the ways that $f(x) = ax^2 + bx + c$ cannot intersect the x-axis.

4. How are the roots of a quadratic equation related to the zeros of a quadratic function?

Focus 4.3

Key Ideas

- A quadratic function may have no zeros, one zero, or two zeros.

No Zeros	One Zero	Two Zeros
		
If the graph of a quadratic function does **not** cross the x-axis, then the function has **no** zeros. If the parabola opens **up**, the vertex must be **above** the x-axis. If the parabola opens **down**, the vertex must be **below** the x-axis.	If the graph of a quadratic function just touches the x-axis, then the function has exactly one zero. In this case, the vertex of the parabola is on the x-axis.	If the graph of a quadratic function crosses the x-axis, then the function has two zeros. If the parabola opens **up**, the vertex must be **below** the x-axis. If the parabola opens **down**, the vertex must be **above** the x-axis.

- You can also find the number of zeros of a quadratic function using the equation of the function.
  - ♦ If the function equation is in factored form, then the zeros are clearly visible.
    - For $f(x) = a(x - p)(x - q)$, there are two zeros at $x = p$ and $x = q$.
    - For $f(x) = a(x - p)^2$, there is one zero at $x = p$.

- If the function is in the vertex form, $f(x) = a(x - h)^2 + k$, then it has two zeros if a and k have opposite signs; no zeros if a and k have the same signs; one zero if $k = 0$.
- If the function is in standard form, $f(x) = ax^2 + bx + c$, then you can use the discriminant $b^2 - 4ac$ to find the number of zeros. The function has two zeros if $b^2 - 4ac > 0$; one zero if $b^2 - 4ac = 0$; no zeros if $b^2 - 4ac < 0$.

- The quadratic equation $ax^2 + bx + c = 0$ has real roots that correspond to the zeros of the function $f(x) = ax^2 + bx + c$. A quadratic equation has no real roots, one real root, or two distinct real roots.

Practise, Apply, Solve 4.3

A

1. Find the number of zeros for each function by sketching its graph.

(a) $f(x) = 2(x - 1.2)^2 - 5$ (b) $f(x) = -1.5(x + 3)^2 + 1.2$

(c) $f(x) = -4(x - 3.6)^2$ (d) $f(x) = 0.5(x - 3.2)^2 + 2$

2. Each of the following functions is in the form $f(x) = a(x - h)^2 + k$. Use the values of a and k to find the number of zeros for each function.

(a) $f(x) = -2(x - 3)^2 - 4$ (b) $f(x) = -3(x + 2.5)^2 + 3.2$

(c) $f(x) = 4(x + 2)^2$ (d) $f(x) = 6.2(x - 3.5)^2 + 4.4$

3. For each of the following functions, which are in standard form, find the value of $b^2 - 4ac$. Use this value to find the number of zeros.

(a) $f(x) = 2x^2 + 3x + 1$ (b) $f(x) = -3x^2 + 5x - 3$

(c) $f(x) = 5x^2 - 4x + 2$ (d) $f(x) = 9x^2 - 14.4x + 5.76$

4. Without solving the equation, find the number of roots for each equation.

(a) $3x^2 - 2x + 1 = 0$ (b) $-2(x - 1.3)^2 + 5 = 0$

(c) $-3(x - 4)(x + 1) = 0$ (d) $-7(x - 4)^2 = 0$

5. Without drawing the graph, say whether the graph of each function intersects the x-axis at one point, intersects the x-axis at two points, or does not intersect the x-axis at all.

(a) $f(x) = x^2 - 6x + 7$ (b) $f(x) = 9 - x^2$

(c) $f(x) = (4 + x)^2$ (d) $f(x) = -2(x - 1)^2 - 1$

B

6. **Knowledge and Understanding:** The demand function for a new product is $p(x) = -4x + 42.5$, where x is the quantity sold in thousands and p is the price in dollars. The company that manufactures the product is planning to buy a new machine for the plant. There are three different types of machine. The cost function for each machine is shown.

Machine A: $C(x) = 4.1x + 92.16$
Machine B: $C(x) = 17.9x + 19.36$
Machine C: $C(x) = 8.8x + 55.4$.

Investigate the break-even quantities for each machine. Which machine would you recommend to the company?

7. For what value(s) of k does the function $f(x) = kx^2 - 4x + k$ have no zeros?

8. The function $g(x) = 4x^2 - 4x + m$ has exactly one zero. What is the value of m?

9. For what values of k does the equation $3x^2 + 4x + k = 0$ have no real roots? one real root? two real roots?

10. The graph of function $f(x) = x^2 - kx + k + 8$ touches the x-axis at one point. What is the value of k?

11. Is it possible for $p^2 + 48$ to equal $-14p$? Explain your answer.

12. **Communication:** Can the graph of a quadratic function cross the x-axis in three different places? Explain your answer.

13. **Application:** The operating costs, y, in thousands of dollars, to produce x personal CD players (x in thousands), at a manufacturing plant are shown in the table. Round the coefficients to one decimal place. If the revenue is \$60 per CD player, at what level of production will the plant break even?

Number of CD Players, *x* (thousands)	2	4	6	8	10	12
Operating Costs, *y* (thousands of dollars)	148	225	287	361	439	521

14. One day, while working on a downtown construction site, Eugenia accidentally dropped her cell phone into the large excavation on the site. Fortunately, the cell phone landed on a pile of soft earth and was not damaged. She asked Tony, the construction manager, who was working nearby to throw the cell phone back up to her.

The height of an object thrown vertically up is modelled by

$$s(t) = s_0 + v_0 t - \frac{1}{2}gt^2$$

where s_0 is the initial height above the ground, v_0 is the initial velocity, g is the acceleration due to gravity, t is the time, and $s(t)$ is the height above the ground at time t.

If Tony is 10 m *below* ground level, and the acceleration due to gravity is 9.8 m/s^2, with what initial velocity must he throw the cell phone for it to reach Eugenia? Explain any assumptions you make and show your calculations, graphs, or both to justify your answer.

15. **Thinking, Inquiry, Problem Solving**

(a) The line $y = x - 1$ intersects the circle $x^2 + y^2 = 25$ at two points. This line is called a **secant**. Find the coordinates of the two points of intersection.

(b) For what value(s) of k will the line $y = x + k$ be a tangent to the circle $x^2 + y^2 = 25$? A **tangent** is a line that touches a circle at exactly one point.

16. **Check Your Understanding:** Suppose you are given a quadratic revenue function and a linear cost function. How could you find the break-even point(s) graphically? algebraically? What solutions might you expect? Give examples and explain any connections between the algebra and the graphs.

C

17. Investigate the number of zeros for the function $f(x) = (k + 1)x^2 + 2kx + k - 1$ for different values of k. For what values of k will the function have no zeros? one zero? two zeros?

18. The area of a right triangle is 15 cm^2 and the hypotenuse is 9 cm long. Find the lengths of the other two sides. (**Hint:** If you know $x^2 + y^2$ and xy, then you can find $(x + y)^2$ and $(x - y)^2$.)

The Chapter Problem — Fundraising

Apply what you have learned in this section to answer these questions about the Chapter Problem on page 300.

CP5. Use the data in the first table on page 300 to write a cost function for each supplier.

CP6. Use your revenue function from CP3 and the cost function for the first supplier in question CP5 to find the corresponding profit function.

CP7. Graph your profit function from question CP6. State the domain and determine the break-even points.

CP8. Repeat questions CP6 and CP7 for the other suppliers.

CP9. Which supplier would you recommend?