4.2 Maximum and Minimum Values of Quadratic Functions

Companies that manufacture and sell items must do research before and after producing an item. Some of this research relates to manufacturing and its costs. Other research concerns, for example, the demand for the item, the selling price, and consumer preferences. While shopping in a mall, you may have been approached by someone conducting "market research." Market research is one way of collecting data.

Data are collected through research and this data may be analyzed in different ways. Most companies want to maximize their profits, so some of the analysis is used to estimate, or model, a profit function. Profit functions are typically quadratic, and you will see why in the following simplified example.

Part 1: Changing a Quadratic Profit Function from Standard Form to Vertex Form by Completing the Square

Think, Do, Discuss

1. The selling price of an item depends, in part, on how many items the company expects to sell. The relation between the price of an item, p, and the number of items sold, x, is called the **demand function**, $p(x)$. Through research, RAYZ Manufacturing collected the following data, where x is the number of pairs of sunglasses sold in thousands and p is the price of one pair in dollars.

Pairs of Sunglasses Sold, x (thousands)	0.7	1.4	1.8	2.1	2.9	3.2
Price, p ($)	30.00	27.50	25.00	22.50	20.00	17.50

   Describe the relation between the price and the number of items sold. How does the price change as the number of items sold increases? Does this change make sense to you? Explain your answer.

2. Enter the data into a graphing calculator. Then perform a linear regression to obtain the demand function, $p(x)$. Round the coefficients to whole numbers.

3. Write the **revenue function**, $R(x)$, based on your demand function. Recall that the revenue is the income from sales.

$$\begin{aligned} R(x) &= \text{number of pairs of glasses sold} \times \text{price of each item} \\ &= x \times \text{demand function} \\ &= x \times p(x) \\ &= ? \end{aligned}$$

4. Research for this item shows that the **cost function** is $C(x) = 3x + 35$. Obtain the **profit function** using $P(x) = R(x) - C(x)$. Simplify the right side by expanding and then collecting like terms. What kind of function is the profit function? Describe the graph of $y = P(x)$.

5. Complete the square of the profit function to determine how many pairs of glasses should be sold to maximize profits.

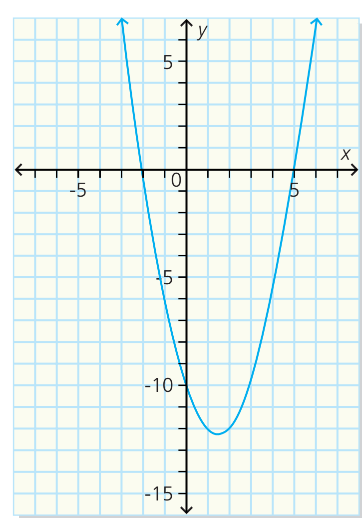
6. Sketch the graph of the profit function. What are the coordinates of the y-intercept? Which form of the equation, standard form or vertex form, did you use to find the y-intercept? Explain the significance of the x-intercepts. For the graph of a profit function, the x-intercepts are called the **break-even points**. Explain this description.

7. Use the profit function, along with the quadratic formula, to find the break-even quantities. Remember that x is in thousands.

8. Use a graphing calculator to check your graph. Check the vertex coordinates and the break-even points using the maximum and zero operations in the **CALC** menu.

Part 2: Finding the Vertex without Completing the Square

Since different forms of a quadratic function are useful for different reasons, you must be able to change from one form to another using algebra. As you have seen, working with decimals and fractions makes completing the square difficult. However, if you are dealing only with quadratic functions, then there is a way to avoid the complicated algebra.



Think, Do, Discuss

1. In Chapter 3, you did some work on transforming functions. If you apply a vertical translation to a parabola, how do the coordinates of the vertex change? Consider the parabola shown here. Its equation is $y = a(x + 2)(x - 5)$. What is the x-coordinate of the vertex? How do you know what the x-coordinate is?

2. Use the coordinates of the y-intercept to find the value of a. Then expand the equation of the parabola so that it is in the form $y = ax^2 + bx + c$.

3. What vertical translation would make the parabola pass through the origin? What is the equation of the translated parabola?

4. Express the new equation in factored form. What are the new x-intercepts? Why is the x-coordinate of the vertex for the new parabola the same as that for the original parabola? Use these new x-intercepts to explain your answer.

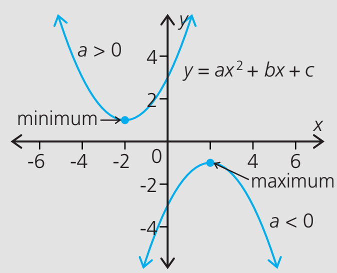
5. Now consider the graph of the profit function, $P(x) = -5x^2 + 31x - 35$, from Part 1. What vertical translation would make the graph pass through the origin? What is the equation of the translated parabola?

6. Express the new equation in factored form. Where are the new x-intercepts? What is the x-coordinate of the vertex? Use the original equation to find the y-coordinate of the vertex for the graph of the profit function. Compare your answer to your result in Part 1.

Focus 4.2

Key Ideas

- The graph of a quadratic function is a parabola.
- If $a > 0$, the parabola opens up and the minimum value of the function occurs at the vertex. If $a < 0$, the parabola opens down and the maximum value of the function occurs at the vertex.
- There are three main forms for the equation of a quadratic function.



  - ♦ In standard form, $f(x) = ax^2 + bx + c$, the y-intercept, $(0, c)$, is clearly visible.
  - ♦ In factored form, $f(x) = a(x - p)(x - q)$, the x-intercepts, $(p, 0)$ and $(q, 0)$, are clearly visible.
  - ♦ In vertex form, $f(x) = a(x - h)^2 + k$, the coordinates of the vertex, (h, k), are clearly visible, and the maximum or minimum value of the function is k.
- You can change the equation of a quadratic function from standard form to vertex form by **completing the square**.
- The parabola for $f(x) = ax^2 + bx$ is a vertical translation of the parabola for $g(x) = ax^2 + bx + c$. For the factored form, $f(x) = x(ax + b)$, the midpoint between the x-intercepts, $(0, 0)$ and $\left(-\frac{b}{a}, 0\right)$, is the x-coordinate of the vertex, $-\frac{b}{2a}$, for parabolas for both $f(x)$ and $g(x)$. Find the y-coordinate of the vertex for $g(x) = ax^2 + bx + c$ by substituting $-\frac{b}{2a}$ for x in the equation. Therefore, for any function $g(x) = ax^2 + bx + c$, the vertex will have the coordinates $\left[-\frac{b}{2a}, g\left(-\frac{b}{2a}\right)\right]$.

Example 1

Research for a given orchard has shown that, if 100 pear trees are planted, then the annual revenue is $90 per tree. If more trees are planted they have less room to grow and generate fewer pears per tree. As a result, the annual revenue per tree is reduced by $0.70 for each additional tree planted. No matter how many trees are planted, the cost of maintaining each tree is $7.40 per year. How many pear trees should be planted to maximize the profit from the orchard for one year?

Solution

Let x represent the number of additional trees planted.

The number of trees in the orchard is $100 + x$, and the annual revenue per tree, in dollars, is $90 - 0.70x$.

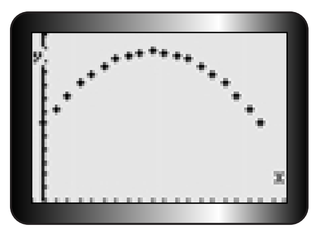
Let $R(x)$ be the total annual revenue from all the trees.

$$\begin{aligned} R(x) &= (100 + x)(90 - 0.70x) \\ &= 9000 - 70x + 90x - 0.70x^2 \\ &= 9000 + 20x - 0.70x^2 \end{aligned}$$

Let $C(x)$ be the cost function. It costs $7.40 to maintain each tree, so the cost of maintaining all the trees in the orchard is $7.40(100 + x), or $C(x) = 740 + 7.4x$.

profit = revenue − cost

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (9000 + 20x - 0.70x^2) - (740 + 7.4x) \\ &= -0.70x^2 + 12.6x + 8260 \qquad \text{Complete the square.} \\ &= -0.70(x^2 - 18x) + 8260 \\ &= -0.70(x^2 - 18x + 81 - 81) + 8260 \\ &= -0.70(x^2 - 18x + 81) + 56.7 + 8260 \\ &= -0.70(x - 9)^2 + 8316.7 \end{aligned}$$



The graph of the profit function is a parabola that opens down. The profit reaches a maximum value at the vertex, (9, 8316.7). To maximize profit, nine additional trees must be planted. A total of 109 pear trees should be planted in the orchard to maximize profit.

The variable x in this solution is discrete because x represents the number of additional trees, which must be a whole number. A graph of $P(x)$ is shown.

Example 2

The demand function for a new product is $p(x) = -5x + 39$, where p represents the selling price of the product and x is the number sold in thousands. The cost function is $C(x) = 4x + 30$.

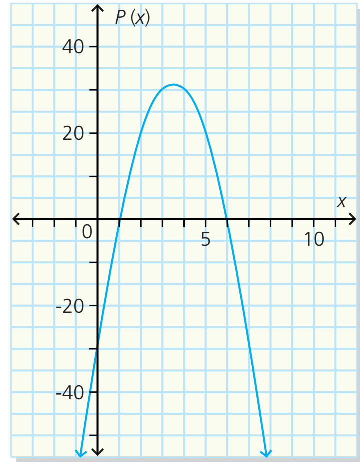
(a) How many items must be sold for the company to break even?

(b) What quantity of items sold will produce the maximum profit?

Solution

profit = revenue − cost

$$\begin{aligned} P(x) &= x \times p(x) - C(x) \\ &= x(-5x + 39) - (4x + 30) \\ &= -5x^2 + 39x - 4x - 30 \\ &= -5x^2 + 35x - 30 \\ &= -5(x^2 - 7x + 6) \\ &= -5(x - 1)(x - 6) \end{aligned}$$



(a) The company breaks even if revenue equals cost, or $P(x) = 0$. Find the zeros, or x-intercepts, for $P(x)$. The factored form of the function shows that the x-intercepts are 1 and 6. The company will break even if 1000 items or 6000 items are sold, since x is the number of items sold in thousands.

(b) The maximum value of the profit function is at the vertex. The x-value of the vertex is the quantity that will produce maximum profit. The vertex lies on the axis of symmetry for the function, halfway between the two x-intercepts. The x-value halfway between 1 and 6 is $\frac{1 + 6}{2} = 3.5$. Profit will be maximized if 3500 items are sold.

Example 3

The cost per hour of running a bus between Burlington and Toronto is modelled by the function $C(x) = 0.0029x^2 - 0.48x + 142$, where x is the speed of the bus in kilometres per hour, and the cost, C, is in dollars. Determine the most cost-efficient speed for the bus and the cost per hour at this speed.

Solution

At the most cost-efficient speed, the hourly cost is a minimum value.

To find this speed and corresponding hourly cost, find the coordinates of the vertex. Instead of completing the square, consider a vertical translation of the cost function.

The value of x that minimizes $0.0029x^2 - 0.48x + 142$ is the same as the value of x that minimizes $0.0029x^2 - 0.48x$ or $x(0.0029x - 0.48)$.

The x-intercepts for the translated function are 0 and $\frac{0.48}{0.0029} \doteq 165.52$. The x-coordinate of the vertex is halfway between these values at $x = 0 + \frac{165.52}{2}$ or about 82.76. The most cost-efficient speed for the inter-city bus is about 83 km/h.

The cost per hour at this value is $C(82.76) = 0.0029(82.76)^2 - 0.48(82.76) + 142$
$\doteq 122.14.$

The hourly cost at this speed is about \$122.14.

Example 4

A farming cooperative collected data showing the effect of different amounts of fertilizer, x, in hundreds of kilograms per hectare (kg/ha), on the yield of carrots, y, in tonnes (t).

(a) Given the data in the table, predict the maximum yield of carrots.

Fertilizer, x (kg/ha)	0	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
Yield, y(t)	0.16	0.46	0.63	0.91	0.96	1.08	1.05	0.88	0.78

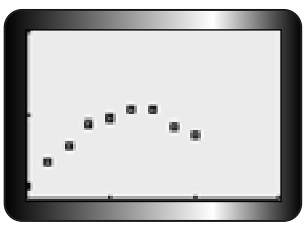
(b) Enter the data into a graphing calculator and use quadratic regression to estimate y as a function of x.

(c) How much fertilizer would you recommend the farmers use? Explain.

Solution

(a) From the table, the maximum yield is about 1.08 t for 125 kg of fertilizer.

(b) Enter the data into lists and create a scatter plot.



Use quadratic regression and store the resulting function in **Y1**. The function relating x and y is $y \doteq -0.53x^2 + 1.38x + 0.14$. (Coefficients are rounded to two decimal places.)

(c) Trace along the graph or choose **4:maximum** from the **CALC** menu to find the coordinates of the vertex.

The coordinates of the vertex of the fitted parabola are about (1.30, 1.04). To maximize the yield of carrots, the farmers should use 130 kg of fertilizer per hectare.

Practise, Apply, Solve 4.2

A

1. **i.** Complete the square to find the coordinates of the vertex for the graph of each function.
ii. State whether each function has a maximum or a minimum value.
iii. Determine the value of x that produces the maximum or minimum value.
iv. State the maximum or minimum value.

(a) $f(x) = x^2 - 5x - 8$ (b) $f(x) = -2x^2 + 16x - 17$

(c) $f(x) = -4x^2 - 6x - 5$ (d) $f(x) = -\frac{1}{2}x^2 + 2x - 9$

2. Without changing the form of the equation, find the coordinates of the vertex for the graph of each function and then sketch the graph.

(a) $f(x) = (x - 3)(x - 8)$ (b) $f(x) = -2(x - 1)(x + 2)$

(c) $f(x) = -0.7(1.1 - x)(3.2 + x)$ (d) $f(x) = -\frac{5}{6}\left(x - \frac{2}{3}\right)\left(x + \frac{1}{6}\right)$

3. Without completing the square, find the coordinates of the vertex and state whether the function has a maximum or minimum value.

(a) $f(x) = 2x^2 - 6x$ (b) $h(t) = -5t^2 + 15t$

(c) $s(t) = -4.9t^2 + 19.6t + 22$ (d) $f(x) = 2.5x^2 - 15x - 24$

4. For each of the following demand functions, x is the number of items sold in thousands and $p(x)$ is the price of each item. Determine
i. the revenue function **ii.** the maximum revenue

(a) $p(x) = -x + 7$ (b) $p(x) = -3x + 11$ (c) $p(x) = -0.4x + 14$

5. For each pair of revenue and cost functions, determine
i. the profit function **ii.** the value of x that maximizes profit

(a) $R(x) = -x^2 + 24x$, $C(x) = 12x + 28$

(b) $R(x) = -2x^2 + 32x$, $C(x) = 14x + 45$

(c) $R(x) = -3x^2 + 26x$, $C(x) = 8x + 18$

(d) $R(x) = -2x^2 + 25x$, $C(x) = 3x + 17$

B

6. **Knowledge and Understanding:** The demand function for a new product is $p(x) = -5x + 21$, where x represents the number of items in thousands and p represents the price in dollars. The cost function is $C(x) = 4x + 14$.

(a) State the corresponding revenue function.

(b) Determine the corresponding profit function.

(c) Complete the square to determine the value of x that will maximize profits.

(d) Find the break-even quantities.

(e) Sketch the graph of the profit function.

7. The cost per hour of running an assembly line in a manufacturing plant is a function of the number of items produced per hour. The cost function is $C(x) = 0.28x^2 - 0.7x + 1$, where $C(x)$ is the cost per hour in thousands of dollars and x is the number of items produced per hour in thousands. Determine the most economical production level.

8. A T-ball baseball player hits a baseball from a tee that is 0.6 m tall. The flight of the ball can be modelled by $h(t) = -4.9t^2 + 6t + 0.6$, where h is the height in metres and t is the time in seconds. When does the ball reach its maximum height? Determine the maximum height of the ball. Answer to one decimal place.

9. The sum of two numbers is 10. What is the largest product of these numbers?

10. Prove that the value of $2x^2 - 8x + 9$ cannot be less than 1.

11. **Communication:** Suppose that $f(x)$ and $g(x)$ are both quadratic functions with graphs that open down. Explain how you know what type of function $y = f(x) + g(x)$ must be. In which direction does the graph open? Where is the maximum point of this function in relation to the maximum points for the first two functions? Explain your answers.

12. (a) An orchard owner has maintained records that show that, if 25 apple trees are planted in one acre, then each tree yields an average of 500 apples.The yield decreases by 10 apples per tree for each additional tree that is planted. How many trees should be planted for maximum total yield?

(b) The cost of maintaining each tree is \$6.50 and the owner can expect to sell his apples for 15¢ each. How many trees should he plant for maximum revenue?

13. Through research, Pizza Pizzazz obtained the demand data in the table, where x represents the number of large pizzas sold per month, and p is the price they plan to charge per pizza.

Number of Pizzas Sold, *x* (thousands)	4.7	5.8	7.3	8.4	8.8	9.8
Price, *p* (\$)	20	18	16	14	12	10

(a) Enter the data into a graphing calculator. Perform a linear regression to obtain a demand function, that relates the price to the number of pizzas sold. Round the coefficients to whole numbers.

(b) Write the revenue function, based on your demand function.

(c) Research also shows that the cost function is $C(x) = 9x + 44$. Determine a profit function and use it to find how many pizzas should be sold each month to maximize profits.

14. Students at an agricultural school collected data showing the effect of different annual amounts of water—rainfall, plus irrigation—x, in hectare-metres (ha-m), on the yield of broccoli, y, in hundreds of kilograms per hectare (100 kg/ha).

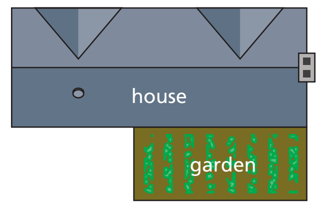
(a) Given the data in the table, predict the maximum yield of broccoli.

Rainfall, x (ha-m)	0.30	0.45	0.60	0.75	0.90	1.05	1.20	1.35	1.50
Yield, y (100 kg/ha)	35	104	198	287	348	401	427	442	418

(b) Use quadratic regression on a graphing calculator to estimate y as a function of x.

(c) What is the optimal annual amount of water? Under these conditions, what is the expected yield of broccoli?

15. Vitaly and Jen have 24 m of fencing to enclose a vegetable garden at the back of their house. What are the dimensions of the largest rectangular garden they could enclose with this length of fencing?



16. A 135-kg steer gains 3.5 kg/day and costs 80¢/day to keep. The market price for beef cattle is $1.65/kg, but the price falls by 1¢/day. When should the steer be sold to maximize profit?

17. **Thinking, Inquiry, Problem Solving:** A family of functions is described by $y = ax^2 + bx$. The graphs of the functions are all parabolas, and each parabola passes through the origin. How do you know that this last sentence is true given the equation? If a is fixed and b varies, then the graphs of these parabolas would still pass through the origin, but the graphs are also related in another way.

(a) Choose any positive, or negative, value for a and create eight equations of the form $y = ax^2 + bx$ by choosing different values for b.

(b) Find the coordinates of the vertex for each of the eight parabolas.

(c) All of the eight vertices are related. Find the relation.

(d) Repeat (a), (b), and (c) for a different value of a.

(e) Write a brief report and include, if possible, a general case.

18. **Check Your Understanding**

(a) What are the three main forms for the equation of a quadratic function?

(b) Provide an example of a quadratic function in each of these three forms.

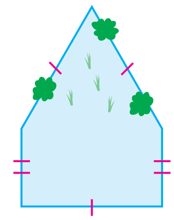
(c) Explain the significance of each form and when it should be used.

(d) Give practical examples of the use of each form.

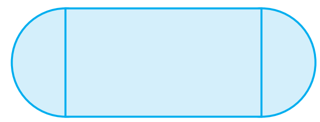
C

19. A rock is thrown straight up in the air, from an initial height, h_0, in metres, with initial velocity, v_0, in metres per second. The height in metres above the ground after t seconds is given by $h(t) = -4.9t^2 + v_0t + h_0$. Find an expression for the time it takes the rock to reach its maximum height.

20. **Application:** Mark is designing a pentagonal-shaped play area for a daycare facility. He has 30 m of nylon mesh to enclose the play area. The triangle in the diagram is equilateral. Find the dimensions of the rectangle plus the triangle, to the nearest tenth of a metre, that will maximize the area he can enclose for the play area.



21. The diagram of a practice field shows a rectangle with a semicircle at each end. The track-and-field coach wants two laps around the field to be 1000 m. The physical education department needs a rectangular field that is as large as possible.



(a) Determine the dimensions of the track that will maximize the entire enclosed area. Do these dimensions meet the needs of the track coach and the physical education department? Explain.

(b) If only the rectangular portion of the field is maximized, can the track team run the 100-m dash along a straight part of the track? Justify your answer.

The Chapter Problem—Fundraising

Apply what you have learned in this section to answer these questions about the Chapter Problem on page 300.

CP1. Use the survey data on page 300 and the total number of students to estimate the number of T-shirts that should sell at each price.

CP2. Enter your data from CP1 into a graphing calculator. Do a linear regression to obtain a demand function relating price to the number of T-shirts sold. Round the coefficients to two decimal places.

CP3. Write the revenue function, based on your demand function from CP2.

CP4. Graph the revenue function and describe its shape. What is its domain?