Lesson 56 – Probability of Dependent Events – Conditional Probabilty

Math 2 Honors - Santowski

Lesson Objectives

- Give the definition of the term **dependence**.
- Explain the meaning of P(E|F).
- Use the properties in this section to compute probabilities.
- Use given information to find probabilities.
- Use the definition of independence to determine whether or not two events are independent.

(A) Review – Independent Events

- If A and B are two independent events, then the probability of occurrence of A and B together is given by:

$$\mathbf{Pr[A \cap B] = Pr[A] \times Pr[B]}$$

- If two fair dice are thrown, the probability of a double 5 is (1/6)(1/6) = 1/36.

(B) Dependent Events

- If two events are **DEPENDENT** ➔ the occurrence of one ***affects the probability*** of the occurrence of the other.
- If the events are dependent, **conditional probability** must be used ➔ which we define as the probability of one event, *A,* occurring given that another, *B,* is already known to have occurred:

(B) Dependent Events

- If the events are **dependent**, conditional probability must be used.
- The conditional probability of B given that A occurs, or on condition that A occurs, is written P[B | A].
- Conditional probability can be found by considering only those events which meet the condition, which in this case is that A occurs.

(B) Dependent Events – Multiplication Rule

- The multiplication rule for the occurrence of both A and B together when they are **NOT independent** is the product of the probability of one event and the conditional probability of the other.

$$\mathbf{Pr[A \cap B] = Pr[A] \times Pr[B \mid A]}$$

$$\mathbf{Pr[A \cap B] = Pr[B] \times Pr[A \mid B]}$$

(B) Dependent Events – Multiplication Rule

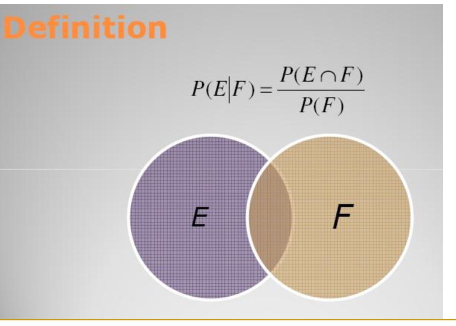
- The formula(s)

$$\Pr[A \cap B] = \Pr[A] \times \Pr[B \mid A]$$
$$\Pr[A \cap B] = \Pr[B] \times \Pr[A \mid B]$$

- Can be rearranged as follows:

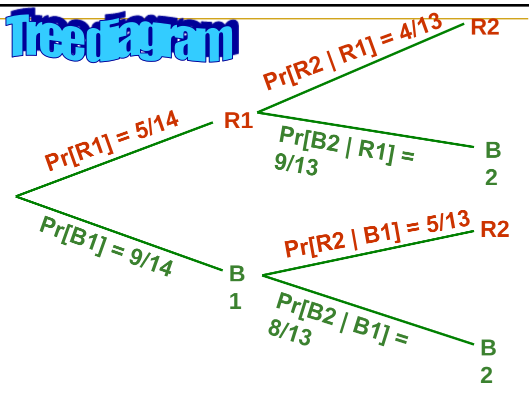
$$\Pr[B \mid A] = \frac{\Pr[A \cap B]}{\Pr[A]}$$
$$\Pr[A \mid B] = \frac{\Pr[A \cap B]}{\Pr[B]}$$

(B) Dependent Events



(C) Example

- There are 14 balls in a box of which 5 are red and 9 are blue. If two balls are selected at random without replacement,
- (i) what is the probability that just one ball is red?
- (ii) What is the probability that both balls are red?



$$\Pr[R1 \cap R2] = \Pr[R1] \times \Pr[R2 \mid R1] = (5/14)(4/13) = 20/182$$
$$\Pr[R1 \cap B2] = \Pr[R1] \times \Pr[B2 \mid R1] = (5/14)(9/13) = 45/182$$
$$\Pr[B1 \cap R2] = \Pr[B1] \times \Pr[R2 \mid B1] = (9/14)(5/13) = 45/182$$
$$\Pr[B1 \cap B2] = \Pr[B1] \times \Pr[B2 \mid B1] = (9/14)(8/13) = 72/182$$

Example (Solution)

- What is the probability that just one ball is red?

$$\Pr[R1 \cap B2] + \Pr[B1 \cap R2] = 45/182 + 45/182 = 90/182$$

- What is the probability that both balls are red?

$$\Pr[R1 \cap R2] = \Pr[R1] \times \Pr[R2 \mid R1] = (5/14)(4/13) = 20/182$$

Example 2

- Three balls are drawn one after the other from a bag containing 6 red balls, 5 yellow balls and 3 green balls. What is the probability that all three balls are yellow if:
- (a) the ball is replaced after each draw and the contents are well mixed?
- (b) the ball is not replaced after each draw?

Example 2

- If the ball is replaced after each draw, then the events are independent and the probability of drawing a yellow ball remains same.
- Pr[Y1] = Pr[Y2] = Pr[Y3] = 5/14
- Pr[three balls are yellow] = (5/14)(5/14)(5/14) = 0.046

Example 2

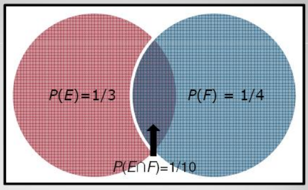
- If the ball is not replaced, then the events are NOT independent and the probability of drawing a yellow ball depends on the previous draw.
- Pr[Y1] = 5/14 and Pr[Y2 | Y1] = 4/13
- And:

$$\Pr[Y3 \mid Y2 \cap Y1] = 3/12$$

$$\Pr[Y3 \cap Y2 \cap Y1] = \left(\frac{5}{14}\right)\left(\frac{4}{13}\right)\left(\frac{3}{12}\right) = 0.027$$

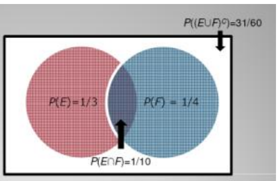
Example 3

- Suppose that $P(E) = 1/3$, $P(F) = 1/4$, and $P(E \cap F) = 1/10$
- Find
  - $P(E|F)$
  - $P(F|E)$
  - $P(E^C|F)$
  - $P(F^C|E)$
  - $P(E^C|F^C)$



Example 3 - Solution

Example 3

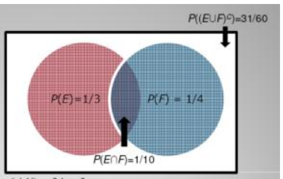


$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{1/10}{1/4} = \frac{4}{10} = \frac{2}{5}$$

$$P(F|E) = \frac{P(F \cap E)}{P(E)} = \frac{1/10}{1/3} = \frac{3}{10}$$

Example 3 - Solution

Example 3



$$P(E^C|F) = \frac{P(E^C \cap F)}{P(F)} = \frac{1/4 - 1/10}{1/4} = \frac{6/40}{1/4} = \frac{24}{40} = \frac{3}{5}$$

$$P(F^C|E) = \frac{P(F^C \cap E)}{P(E)} = \frac{1/3 - 1/10}{1/3} = \frac{7/30}{1/3} = \frac{21}{30} = \frac{7}{10}$$

$$P(E^C|F^C) = \frac{P(E^C \cap F^C)}{P(F^C)} = \frac{31/60}{1 - 1/4} = \frac{31/60}{3/4} = \frac{124}{180} = \frac{31}{45}$$

Example 4

- Given 2 die, What is the probability of rolling the sum of 3, given that we rolled a sum less than 5

Example 4 - Solution

- Event E is rolling the sum of 3 ➔ E = {(1,2), (2,1)} and we know P(E) = 2/36 = 1/18
- Event F is rolling a sum less than 5 ➔ F = {(1,1), (1,2), (2,1), (1,3), (3,1), (2,2)}
- **Therefore, P(E|F) = 2/6 = 1/3**
- or using our formula

$$p(E\mid F)=\frac{p(E\cap F)}{p(F)}=\frac{2/36}{6/36}=\frac{2}{6}=\frac{1}{3}$$

Example 5

- The probability that event E occurs is 0.63
- The probability that event F occurs is 0.45
- The probability that both events occur is 0.10
-
- Find P(E|F) and P(F|E)

Example 5 - Solution

- The probability that event E occurs is 0.63
- The probability that event F occurs is 0.45
- The probability that both events occur is 0.10
-
- Find P(E|F) and P(F|E)

$$p(E\mid F)=\frac{p(E\cap F)}{p(F)}=\frac{0.1}{0.45}\approx 0.222$$

$$p(E\mid F)=\frac{p(E\cap F)}{p(F)}=\frac{0.1}{0.63}\approx 0.159$$

Example 6

- Three manufacturing plants *A*, *B*, and *C* supply 20%, 30%, and 50% respectively of all shock absorbers used by a certain car manufacturer.
- Records show that the percentage of defective items produced by plants *A*, *B*, and *C* is 3%, 2%, and 1% respectively
- What is the probability that a randomly chosen shock absorber
  - came from plant A and was defective?
  - came from plant B and was not defective?
  - was defective?

Example 6 - Solution

- What is the probability that a randomly chosen shock absorber
  - came from plant A and was defective?
    - *P*(*A*∩*D*) = *P*(*D*|*A*)•*P*(*A*) = (.03)(.2)=.006 = .6%
  - came from plant B and was not defective?
    - *P*(*B*∩*ND*) = *P*(*ND*|*B*)•*P*(*B*) = (.98)(.3)=.294 = 29.4%
  - was defective?
    - *P*((*A*∩*D*)∪(*B*∩*D*)∪(*C*∩*D*)) = *P*(*A*∩*D*)+*P*(*B*∩*D*)+*P*(*C*∩*D*)
      = *P*(*D*|*A*)•*P*(*A*) + *P*(*D*|*B*)•*P*(*B*)+ *P*(*D*|*C*)•*P*(*C*)
      = (.03)(.2)+(.02)(.3)+(.01)(.5) = .017 = 1.7%
  - *D* = defective, *ND* = not defective

Independent Events

- Two events are <u>independent</u> if knowing one has occurred does not affect the probability of the other
  - If E and F are independent events
    - $P(E) = P(E|F)$ and $P(F)=P(F|E)$
    - $P(E \cap F) = P(E|F) \cdot P(F) = P(E) \cdot P(F)$
  - If $P(E \cap F) = P(E) \cdot P(F)$, then E and F are independent events
  - If two events are not independent, they are said to be dependent.

Independent Events - Examples

You roll two fair die, one is green and the other is red, and observe the outcomes

- Let A be the event that their sum is 7
- Let B be the event that the red die shows an even number
- Are these events independent?
- Are these events mutually exclusive?

Independent Events – Examples (Solns)

- You roll two fair die, one is green and the other is red, and observe the outcomes
  - Let A be the event that their sum is 7
    - $A=\{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$
    - $P(A) = 6/36 = 1/6$
  - Let B be the event that the red die shows an even number
    - $B=\{(1,2), (2,2), (3,2), (4,2), (5,2), (6,2), (1,4), (2,4), (3,4), (4,4), (5,4), (6,4), (1,6), (2,6), (3,6), (4,6), (5,6), (6,6)\}$
    - $P(B) = 18/36 = 1/2$
  - Are these events independent? YES
    - $P(A \cap B) = 3/36 = 1/12$
    - $P(A) \cdot P(B) = (1/6)(1/2) = 1/12$

Independent Events – Examples (Solns)

You roll two fair die, one is green and the other is red, and observe the outcomes

- Let A be the event that their sum is 4
- Let B be the event that the red die shows an even number
- Are these events independent?

Independent Events – Examples (Solns)

- You roll two fair die, one is green and the other is red, and observe the outcomes
  - Let A be the event that their sum is 4
    - $A=\{(1,3), (2,2), (3,1)\}$
    - $P(A) = 3/36 = 1/12$
  - Let B be the event that the red die shows an even number
    - $B=\{(1,2), (2,2), (3,2), (4,2), (5,2), (6,2), (1,4), (2,4), (3,4), (4,4), (5,4), (6,4), (1,6), (2,6), (3,6), (4,6), (5,6), (6,6)\}$
    - $P(B) = 18/36 = 1/2$
  - Are these events independent? NO
    - $P(A \cap B) = 1/36$
    - $P(A) \cdot P(B) = (1/12)(1/2) = 1/24$

Homework