Lesson 45 - Trigonometric Identities

Math 2 Honors - Santowski

(A) Review of Equations

- An equation is an algebraic statement that is true for only several values of the variable
- The linear equation $5 = 2x - 3$ is only true for …..?
- The quadratic equation $0 = x^2 - x - 6$ is true only for ……?
- The trig equation $\sin(\theta) = 1$ is true for ……?
- The reciprocal equation $2 = 1/x$ is true only for ….?
- The root equation $4 = \sqrt{x}$ is true for …..?

(A) Review of Equations

- An equation is an algebraic statement that is true for only several values of the variable
- The linear equation $5 = 2x - 3$ is only true for the x value of 4
- The quadratic equation $0 = x^2 - x - 6$ is true only for $x = -2$ and $x = 3$ (i.e. $0 = (x - 3)(x + 2)$)
- The trig equation $\sin(\theta) = 1$ is true for several values like 90°, 450°, -270°, etc…
- The reciprocal equation $2 = 1/x$ is true only for $x = ½$
- The root equation $4 = \sqrt{x}$ is true for one value of x = 16

(B) Introduction to Identities

- Now imagine an equation like $2x + 2 = 2(x + 1)$ and we ask ourselves the same question ➔ for what values of x is it true?
- Now ➔ $4(x - 2) = (x - 2)(x + 2) - (x - 2)^2$ and we ask ourselves the same question ➔ for what values of x is it true?
- What about $\frac{1}{x-1} = \frac{x}{x^2 - x}$

(B) Introduction to Identities

- Now imagine an equation like $2x + 2 = 2(x + 1)$ and we ask ourselves the same question ➔ for what values of x is it true?
- We can actually see very quickly that the right side of the equation expands to $2x + 2$, so in reality we have an equation like $2x + 2 = 2x + 2$
- But the question remains ➔ for what values of x is the equation true??
- Since both sides are identical, it turns out that the equation is true for **ANY** value of x we care to substitute!
- So we simply assign a slightly different name to these special equations ➔ we call them IDENTITIES because they are true for ALL values of the variable!

(B) Introduction to Identities

- For example, $4(x - 2) = (x - 2)(x + 2) - (x - 2)^2$
- Is this an identity (true for ALL values of x) or simply an equation (true for one or several values of x)???
- The answer lies with our mastery of fundamental algebra skills like expanding and factoring ➔ so in this case, we can perform some algebraic simplification on the right side of this equation
- $RS = (x^2 - 4) - (x^2 - 4x + 4)$
- $RS = -4 + 4x - 4$
- $RS = 4x - 8$
- $RS = 4(x - 2)$
- So yes, this is an identity since we have shown that the sides of the "equation" are actually the same expression

(C) Domain of Validity

- When two algebraic expressions (the LS and RS of an equation) are equal to each other for specific values of the variable, then the equation is an **IDENTITY**
- This equation is an identity **as long as the variables make the statement defined**, in other words, as long as the variable is in the domain of each expression.
- The set of all values of the variable that make the equation defined is called the domain of validity of the identity. (It is really the same thing as the domain for both sides of the identity

(C) Basic Trigonometric Identities

- Recall our definitions for sin(θ) = o/h, cos(θ) = a/h and tan(θ) = o/a
- So now one trig identity can be introduced ➔if we take sin(θ) and divide by cos(θ), what do we get?

(C) Basic Trigonometric Identities

- Recall our definitions for sin(θ) = o/h, cos(θ) = a/h and tan(θ) = o/a
- So now one trig identity can be introduced ➔if we take sin(θ) and divide by cos(θ), what do we get?
- $\frac{\sin(\theta)}{\cos(\theta)} = \frac{o/h}{a/h} = \frac{o}{a} = \tan(\theta)$
- So the tan ratio is actually a quotient of the sine ratio divided by the cosine ratio
- What would the cotangent ratio then equal?

(C) Basic Trigonometric Identities

- So the tan ratio is actually a quotient of the sine ratio divided by the cosine ratio
- We can demonstrate this in several ways ➔ we can substitute any value for θ into this equation and we should notice that both sides always equal the same number
- Or we can graph f(θ) = sin(θ)/cos(θ) as well as f(θ) = tan(θ) and we would notice that the graphs were identical
- This identity is called the QUOTIENT identity

(C) Basic Trigonometric Identities

Pythagorean Identities (3)

a) Use your calculator to evaluate $\sin^2 x + \cos^2 x$ for

$x = 30^o$	$x = \frac{\pi}{2}$	$x = 149^o$	$x = -\frac{5\pi}{11}$

b) Conjecture the identity value for $\sin^2 x + \cos^2 x$:

c) State the domains of

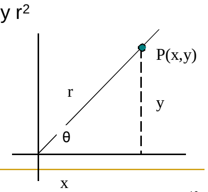
$y = \cos x$	$y = \sin x$	$y = \sin x + \cos x$	$y = \sin^2 x$	$y = \cos^2 x$	$y = \sin^2 x + \cos^2 x$

d) Prove that $\sin^2 x + \cos^2 x = 1$ is an identity and state its domain of validity.

e) Discuss why this identity is called a Pythagorean Identity.

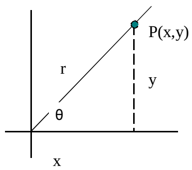
(C) Basic Trigonometric Identities

- Another key identity is called the Pythagorean identity
- In this case, since we have a right triangle, we apply the Pythagorean formula and get $x^2 + y^2 = r^2$
- Now we simply divide both sides by r^2



(C) Basic Trigonometric Identities

- Now we simply divide both sides by r^2
- and we get $x^2/r^2 + y^2/r^2 = r^2/r^2$
- Upon simplifying, $(x/r)^2 + (y/r)^2 = 1$
- But $x/r = \cos(\theta)$ and $y/r = \sin(\theta)$ so our equation becomes
- $(\cos(\theta))^2 + (\sin(\theta))^2 = 1$
- Or rather $\cos^2(\theta) + \sin^2(\theta) = 1$
- Which again can be demonstrated by
- substitution or by graphing



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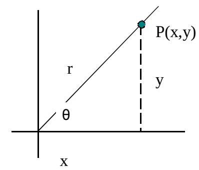
(C) Basic Trigonometric Identities

- Now we simply divide both sides by x^2
- and we get $x^2/x^2 + y^2/x^2 = r^2/x^2$
- Upon simplifying, $1 + (y/x)^2 = (r/x)^2$
- But $y/x = \tan(\theta)$ and $r/x = \sec(\theta)$ so our equation becomes
- $1 + (\tan(\theta))^2 = (\sec(\theta))^2$
- Or rather $1 + \tan^2(\theta) = \sec^2(\theta)$
- Which again can be demonstrated by
- substitution or by graphing



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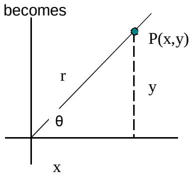
(C) Basic Trigonometric Identities

- Now we simply divide both sides by y^2
- and we get $x^2/y^2 + y^2/y^2 = r^2/y^2$
- Upon simplifying, $(x/y)^2 + 1^2 = (r/y)^2$
- But $x/y = \cot(\theta)$ and $r/y = \csc(\theta)$ so our equation becomes
- $(\cot(\theta))^2 + 1 = (\csc(\theta))^2$
- Or rather $1 + \cot^2(\theta) = \csc^2(\theta)$
- Which again can be demonstrated by
- substitution or by graphing



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(G) Simplifying Trig Expressions

- Simplify the following expressions:

$(a)\ 2-2\cos^2 x$

$(b)\ \sin^2 x\cos x+\cos^3 x$

$(c)\ (\cos x-\sin x)^2$

$(d)\ \dfrac{2-2\cos^2 x}{1+\cos x}$

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(G) Simplifying Trig Expressions

2. Simplify.

(a) $\sin x\left(\frac{1}{\cos x}\right)$ (b) $(\cos x)(\tan x)$ (c) $1-\cos^2 x$

(d) $1-\sin^2 x$ (e) $\cos^2 x+\sin^2 x$ (f) $(1-\sin x)(1+\sin x)$

(g) $\frac{\tan x}{\sin x}$ (h) $\frac{\frac{\sin x}{\cos x}}{\tan x}$ (i) $\left(\frac{1}{\tan x}\right)\sin x$

(j) $\frac{1+\tan^2 x}{\tan^2 x}$ (k) $\frac{\sin x\cos x}{1-\sin^2 x}$ (l) $\frac{1-\cos^2 x}{\sin x\cos x}$

(m) $\frac{1}{\sin x}+\frac{1}{\cos x}$ (n) $\tan x+\frac{1}{\cos x}$ (o) $\frac{1}{\tan x}+\sin x$

3. Factor each expression.

(a) $1-\cos^2\theta$ (b) $1-\sin^2\theta$

(c) $\sin^2\theta-\cos^2\theta$ (d) $\sin\theta-\sin^2\theta$

(e) $\cos^2\theta+2\cos\theta+1$ (f) $\sin^2\theta-2\sin\theta+1$

(E) Examples

- Prove $\tan(x)\cos(x) = \sin(x)$
- Prove $\tan^2(x) = \sin^2(x)\cos^{-2}(x)$
- Prove $\tan x+\dfrac{1}{\tan x}=\dfrac{1}{\sin x\cos x}$
- Prove $\dfrac{\sin^2 x}{1-\cos x}=1+\cos x$

(E) Examples

- Prove tan(x) cos(x) = sin(x)

$$LS = \tan x \cos x$$
$$LS = \frac{\sin x}{\cos x} \cos x$$
$$LS = \sin x$$
$$\therefore LS = RS$$

(E) Examples

- Prove $\tan^2(x) = \sin^2(x) \cos^{-2}(x)$

$$RS = \sin^2 x \cos^{-2} x$$
$$RS = (\sin x)^2 \left(\frac{1}{\cos^2 x}\right)$$
$$RS = (\sin x)^2 \frac{1}{(\cos x)^2}$$
$$RS = \frac{(\sin x)^2}{(\cos x)^2}$$
$$RS = \left(\frac{\sin x}{\cos x}\right)^2$$
$$RS = \tan^2 x$$
$$\therefore RS = LS$$

(E) Examples

- Prove $\tan x + \frac{1}{\tan x} = \frac{1}{\sin x \cos x}$

$$LS = \tan x + \frac{1}{\tan x}$$
$$LS = \frac{\sin x}{\cos x} + \frac{1}{\frac{\sin x}{\cos x}}$$
$$LS = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}$$
$$LS = \frac{\sin x \sin x + \cos x \cos x}{\cos x \sin x}$$
$$LS = \frac{\sin^2 x + \cos^2 x}{\cos x \sin x}$$
$$LS = \frac{1}{\cos x \sin x}$$
$$\therefore LS = RS$$

(E) Examples

- Prove $\frac{\sin^2 x}{1 - \cos x} = 1 + \cos x$

$$LS = \frac{\sin^2 x}{1 - \cos x}$$
$$LS = \frac{1 - \cos^2 x}{1 - \cos x}$$
$$LS = \frac{(1 - \cos x)(1 + \cos x)}{(1 - \cos x)}$$
$$LS = 1 + \cos x$$
$$\therefore LS = RS$$

Examples

- http://teachersites.schooldesk.net/content/4/96/1399/my%20files/3.4%20identities.pdf
- http://www.analyzemath.com/trigonometry_worksheets.html

1. Establish each identity:

(a) $\tan\theta \cot\theta - \sin^2\theta = \cos^2\theta$

(b) $(1 - \cos^2\theta)(1 + \cot^2\theta) = 1$

(c) $4\sin^2\theta + 2\cos^2\theta = 4 - 2\cos^2\theta$

(d) $\frac{\sin\theta}{1 + \cos\theta} + \frac{1 + \cos\theta}{\sin\theta} = 2\csc\theta$

(F) Internet Links

- We will spend class time working through the questions from the following link:
- Webmath.com: Trigonometric Identities
- Likewise, you can try these on-line questions: Introductory Exercises from U. of Sask EMR
- You can watch some on-line videos on proving identities: Proving Identities from MathTV

(D) Solving Trig Equations with Substitutions ➔ Identities

- Solve $\tan\theta \times \cos\theta - 1 = 0 \quad \text{for} -2\pi \le \theta \le 2\pi$

(D) Solving Trig Equations with Substitutions ➔ Identities

- Solve $\tan\theta \times \cos\theta - 1 = 0 \quad \text{for} -2\pi \le \theta \le 2\pi$
- But $\tan x \cos x = \dfrac{\sin x}{\cos x}\cos x$

  So $\tan x \cos x = \sin x$
- So we make a substitution and simplify our equation ➔ $\sin\theta - 1 = 0 \quad \text{for} -2\pi \le \theta \le 2\pi$

  $$\theta = -\frac{3\pi}{2}, \frac{\pi}{2}$$

(E) Examples

- Solve $\dfrac{\sin^2 x}{1-\cos x} = 2 \quad \text{for } -2\pi \le x \le 2\pi$

(E) Examples

- Solve $\dfrac{\sin^2 x}{1-\cos x} = 2 \quad \text{for } -2\pi \le x \le 2\pi$
- Now, one option is:

$$\frac{\sin^2 x}{1-\cos x} = 2 \quad \text{for } -2\pi \le x \le 2\pi$$
$$\frac{1-\cos^2 x}{1-\cos x} = 2$$
$$\frac{(1-\cos x)(1+\cos x)}{1-\cos x} = 2$$
$$1+\cos x = 2$$
$$\therefore \cos x = -1$$
$$x = \pm\pi$$

(E) Examples

- Solve the following

(a) $\sin x + 1 - 2\cos^2 x = 0 \quad \text{for } -2\pi \le x \le 2\pi$

(b) $1 - \sin x = 2\cos^2 x \quad \text{for } -2\pi \le x \le 2\pi$

(c) $\dfrac{1}{\cos x} - \sin x \tan x = -\dfrac{1}{\sqrt{2}} \quad \text{for } -2\pi \le x \le 2\pi$

(F) Example

- Since $1 - \cos^2 x = \sin^2 x$ is an identity, is

$$\sqrt{1-\cos^2 x} = \sin x$$

(F) Homework

- HW
- S14.3, p906-7, Q13-47odds,51,55

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