Lesson 43 – Trigonometric Functions

Math 2 Honors - Santowski

Lesson Objectives

- Graph and analyze a sinusoidal function
- Make the connection between angles in standard position and sinusoidal functions
- Transformations of sinusoidal functions and their features
- Apply sinusoidal functions to periodic phenomenon

Fast Five

- Use your calculator and graph the function f(x) = sin(x) on the domain -720° < x < 720°
- Evaluate sin(50°) ➔ use your graph
- Evaluate sin(130°) ➔ use your graph
- Evaluate sin(230°) ➔ use your graph
- Evaluate sin(320°) ➔ use your graph
- Evaluate sin(765°)➔ No graph nor calculator
- Evaluate sin(-50°) ➔ use your graph

(A) Graph of f(x) = sin(x)

- We can use our knowledge of angles on Cartesian plane and our knowledge of the trig ratios of special angles to create a list of points to generate a graph of f(x) = sin(x)

(A) Graph of f(x) = sin(x)

- We have the following points from the first quadrant that we can graph: (0,0), (30,0.5), (45,0.71), (60,0.87) and (90,1)
- We have the following second quadrant points that we can graph: (120,0.87), (135,0.71), (150,0.5), and (180,0)
- We have the following third quadrant points: (210,-0.50), (225,-0.71), (240,-0.87) and (270,-1)
- Finally we have the 4th quadrant points: (300,-0.87), (315,-.71), (330,-0.5) and (360,0)

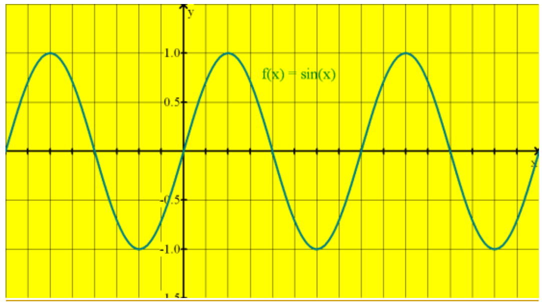
(A) Graph of f(x) = sin(x)

- Now we need to consider the co-terminal angles as well to see what happens beyond our one rotation around the 4 quadrants
- For example, consider that sin(390) is the sine ratio of the first positive coterminal angle with 390-360 = 30 degrees
- So, sin(390) = sin(30) = 0.5
- So we can extend our list of points to include the following:
- (390,0.5), (405,0.71), (420,0.87) and (450,1)
- (480,0.87), (495,0.71), (510,0.5), and (540,0)
- (570,-0.50), (585,-0.71), (600,-0.87) and (630,-1)
- (660,-0.87), (675,-.71), (690,-0.5) and (720,0)

(A) Graph of f(x) = sin(x)

- Now we need to consider the negative angles as well to see what happens by rotating "backwards"
- For example, consider that sin(-30) is the sine ratio of the first negative coterminal angle with 360-30 = 330 degrees
- So, sin(-30) = sin(330) = -0.5
- So we can extend our list of points to include the following:
- (-30,-0.5), (-45,-0.71), (-60,-0.87) and (-90,-1)
- (-120,-0.87), (-135,-0.71), (-150,-0.5), and (180,0)
- (-210,0.50), (-225,0.71), (-240,0.87) and (-270,1)
- (-300,0.87), (-315,.71), (-330,0.5) and (-360,0)

(A) Graph of f(x) = sin(x)



(A) Features of f(x) = sin(x)

- The graph is **periodic (meaning that it repeats itself)**
- **Domain**:
- **Range**:
- **Period**: length of one cycle, how long does the pattern take before it repeats itself ➔.
- **x-intercepts**:
- **Equilibrium axis** or **axis of the curve** ➔
- **amplitude**: max height above equilibrium position - how high or low do you get ➔
- **y-intercept**:
- **max. points**:
- **min. points:**

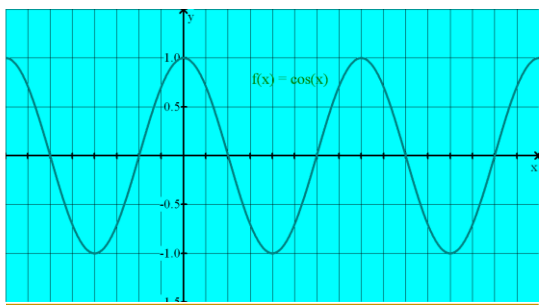
(A) Features of f(x) = sin(x)

- The graph is **periodic (meaning that it repeats itself)**
- **Domain**: x E **R**
- **Range**: [-1,1]
- **Period**: length of one cycle, how long does the pattern take before it repeats itself ➔ 360° or 2 π rad.
- **x-intercepts**: every 180°, x = 180°n where n E I or πn where n E I.
- **Equilibrium axis** or **axis of the curve** ➔ x axis
- **amplitude**: max height above equilibrium position - how high or low do you get => 1 unit
- **y-intercept**: (0°,0)
- **max. points**: 90°+ 360°n (or 2π + 2 π n)
- **min. points:** 270°+ 360°n or -90° + 360°n or -π/2 + 2 π n

(B) Graph of f(x) = cos(x)

- We can repeat the same process of listing points and plotting them to see the graph of f(x) = cos(x)
- Our first quadrant points include:
- (0,1), (30,0.87), (45,0.71), (60,0.5) and (90,0)
- And then we could list all the other points as well, or simply turn to graphing technology and generate the graph:

(B) Graph of f(x) = cos(x)



(B) Features of f(x) = cos(x)

- The graph is **periodic**
- **Domain**:
- **Range**:
- **Period**: length of one cycle, how long does the pattern take before it repeats itself ➔.
- **Equilibrium axis** or **axis of the curve** ➔
- **x-intercepts**:
- **amplitude**: max height above equilibrium position - how high or low do you get ➔
- **y-intercept**:
- **max. points**:
- **min. points:**

(B) Features of f(x) = cos(x)

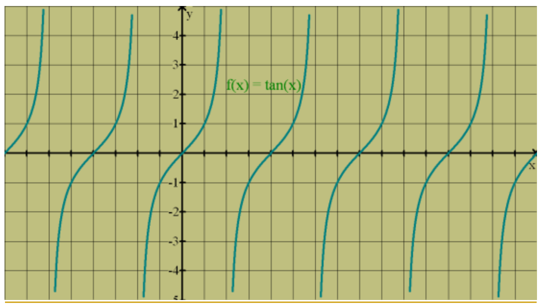
- The graph is **periodic**
- **Domain**: x E **R**
- **Range**: [-1,1]
- **Period**: length of one cycle, how long does the pattern take before it repeats itself ➔ 360° or 2 π rad.
- **x-intercepts**: every 180° starting at 90°, x = 90° + 180°n where n E I (or π/2 + π n where n E I)
- **Equilibrium axis** or **axis of the curve** ➔ x axis
- **amplitude**: max height above equilibrium position - how high or low do you get => 1 unit
- **y-intercept**: (0°,1)
- **max. points**: 0° + 360°n (2 π n)
- **min. points:** 180° + 360°n or -180° + 360°n (or π + 2 π n)

(C) Graph of f(x) = tan(x)

- Likewise, for the tangent function, we list points and plot them:

- (0,0), (30,0.58), (45,1), (60,1.7), (90,undefined)
- (120,-1.7), (135,-1), (150,-0.58), (180,0)
- (210, 0.58), (225,1), (240,1.7), (270,undefined)
- (300,-1.7), (315,-1), (330,-0.58), (360,0)

(C) Graph of f(x) = tan(x)



(C) Features of f(x) = tan(x)

- The graph is **periodic**
- **Domain**:
- **Asymptotes**:
- **Range**:
- **Period**: length of one cycle, how long does the pattern take before it repeats itself ➔
- **x-intercepts**:
- **amplitude**: max height above equilibrium position - how high or low do you get ➔
- **y-intercept**:
- **max. points**:
- **min. points:**

(C) Features of f(x) = tan(x)

- The graph is **periodic**
- **Domain**: x E **R** where x cannot equal 90°, 270°, 450°, or basically 90° + 180°n where n E I
- **Asymptotes**: every 180° starting at 90°
- **Range**: x E **R**
- **Period**: length of one cycle, how long does the pattern take before it repeats itself = 180° or π rad.
- **x-intercepts**: x = 0°, 180°, 360°, or basically 180°n where n E I or x = πn
- **amplitude**: max height above equilibrium position - how high or low do you get => none as it stretches on infinitely
- **y-intercept**: (0°,0)
- **max. points**: none
- **min. points:** none

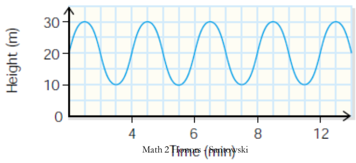
(D) Internet Links

- Unit Circle and Trigonometric Functions sin(x), cos(x), tan(x) from AnalyzeMath
- Relating the unit circle with the graphs of sin, cos, tan from Maths Online

Trigonometric Functions – Sinusoidal Modeling

(A) Fast Five

- The graph shows John's height above the ground as a function of time as he rides a Ferris wheel.
  - **(a)** State the maximum and minimum height of the ride.
  - **(b)** How long does the Ferris wheel take to make one complete revolution?
  - **(c)** What is the amplitude of the curve? How does this relate to the Ferris wheel?
  - **(d)** Determine the equation of the axis of the curve.



(A) Key Terms

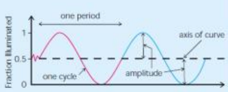
- Define the following key terms that relate to trigonometric functions:
- (a) period
- (b) amplitude
- (c) axis of the curve (or equilibrium axis)

(A) Key Terms

Key Ideas

- Repeating data forms a periodic function.
- A periodic function has a self-repeating graph.
- The cycle of a graph is the smallest complete repeating pattern of the graph.
- The length of one cycle is called the **period**.
- The horizontal line that is halfway between the maximum and minimum values of a periodic curve is called the **axis of the curve**.
- The equation of the axis of the curve is

$$y = \frac{\text{maximum value} + \text{minimum value}}{2}$$



- The magnitude of the vertical distance from the axis of the curve to either the maximum or minimum value is called the **amplitude** of the function. The amplitude, a, is calculated as

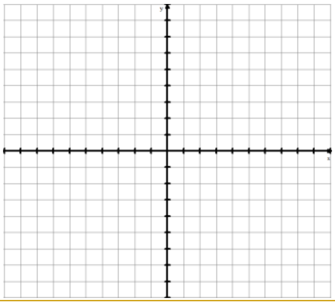
$$a = \frac{\text{maximum value} - \text{minimum value}}{2}$$

(B) Applying the key terms

5. Sketch periodic graphs to satisfy the given properties.

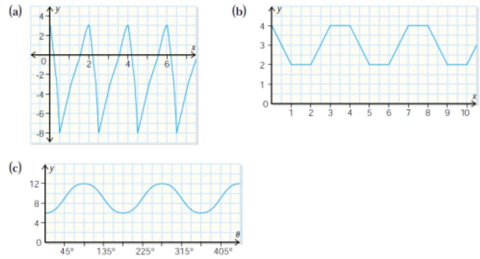
Shape	Period	Amplitude	Equation of Axis	Number of Cycles
	4	6	$y = 2$	2
	3	4	$y = 1$	3
	$\frac{1}{2}$	5	$y = -3$	2

(B) Applying the key terms



(B) Applying the key terms

6. State the period, amplitude, and the equation of the axis for each function.



(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

The height, h, of a basket on a water wheel at time t is given by $h(t) = \sin(6t)°$, where t is in seconds and h is in metres.

(a) How high is the basket at 14 s?

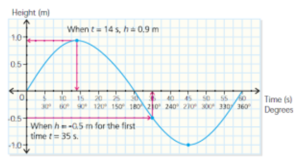
(b) When will the basket first be 0.5 m under water?

(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

(a) The values of h and t can be interpolated from a graph. Prepare a table. In this case, 5-s intervals were used, although the interval size could be different.

t (seconds)	0	5	10	15	20	25	30	35	40	45	50	55	60
$(6t)°$	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
$h(t) = \sin(6t)°$ (metres)	0	0.5	0.87	1	0.87	0.5	0	−0.5	−0.87	−1	−0.87	−0.5	0

Interpolating from the graph gives a value of about 0.9 m, as shown.



(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

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(a) How high is the basket at 14 s?

(b) When will the basket first be 0.5 m under water?

The question could also be answered by substituting into the equation.

Then,

$$h(14) = \sin(6 \times 14)° = \sin 84° \doteq 0.995$$

Substitution gives a more accurate answer in this case. At 14 s, the height is almost 1 m.

A height of 0.5 m under water corresponds to a height of −0.5 m in the model. Therefore,

$$h(t) = \sin(6t)°$$
$$-0.5 = \sin(6t)°$$ Interpolation shows that $\sin 210° = -0.5$.

The basket will be 0.5 m under water for the first time at 35 s.

(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

Application: A buoy rises and falls as it rides the waves. The equation $h(t) = \cos\frac{\pi}{5}t$ models the displacement of the buoy in metres at t seconds.

(a) Graph the displacement from 0 to 20 s, in 2.5-s intervals.

(b) Determine the period of the function from the graph. Determine the period algebraically from the equation.

(c) What is the displacement at 35 s?

(d) At what time, to the nearest second, does the displacement first reach −0.8 m?

(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

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(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

- A spring bounces up and down according to the model $d(t) = 0.5 \cos 2t$, where d is the displacement in centimetres from the rest position and t is the time in seconds. The model does not consider the effects of gravity.
  - **(a)** Make a table for $0 \le t \le 9$, using 0.5-s intervals.
  - **(b)** Draw the graph.
  - **(c)** Explain why the function models periodic behaviour.
  - **(d)** What is the relationship between the amplitude of the function and the displacement of the spring from its rest position?
  - **(e)** What is the period and what does it represent in the context of this question?
  - **(f)** What is the amplitude and what does it represent in the context of this question?

(C) Modeling Periodic Phenomenon & Transformed Sinusoidal Curves

- Since we are dealing with general sinusoidal curves, the basic equation of all our curves should involve f(x) = sin(x) or f(x) = cos(x)
- In our questions, though, we are considering TRANSFORMED sinusoidal functions however ➔ HOW do we know that????
- So our general formula in each case should run something along the lines of f(x) = ***a***sin(***k***(x+***c***)) + ***d***

(D) The General Sinusoidal Equation

- In the equation f(x) = ***a***sin(***k***(x+***c***)) + ***d***, explain what:
- a represents?
- k represents?
- c represents?
- d represents?

(D) The General Sinusoidal Equation

- In the equation f(x) = ***a***sin(***k***(x+***c***)) + ***d***, explain what:
- **a** represents? ➔ vertical stretch/compression ➔ so changes in the **amplitude**
- **k** represents? ➔ horizontal stretch/compression ➔ so changes in the **period**
- **c** represents? ➔ horizontal translations ➔ so changes in the **starting point of a cycle** (phase shift)
- **d** represents?➔ vertical translations ➔ so changes in the **axis of the curve (equilibrium)**

(E) Transforming y = sin(x)

- Graph y = sin(x) as our reference curve
- (i) Graph y = sin(x) + 2 and y = sin(x) – 1 and analyze ➔ what features change and what don't?
- (ii) Graph y = 3sin(x) and y = ¼sin(x) and analyze ➔ what features change and what don't?
- (iii) Graph y = sin(2x) and y = sin(½x) and analyze ➔ what features change and what don't?
- (iv) Graph y = sin(x+π/4) and y = sin(x-π/3) and analyze ➔ what changes and what doesn't?
- We could repeat the same analysis with either y = cos(x) or y = tan(x)

(F) Combining Transformations

- We continue our investigation by graphing some other functions in which we have combined our transformations
- (i) Graph and analyze y = 2 sin 3(x - 60°) + 1 ➔ identify transformations and state how the key features have changed
- (ii) Graph and analyze y = 2 cos [2(x - π/4)] – 3 ➔ identify transformations and state how the key features have changed
- (iii) Graph and analyze y = tan(½ x + π/4) – 3 ➔ identify transformations and state how the key features have changed

(G) Writing Sinusoidal Equations

- ex 1. Given the equation y = 2sin3(x - 60°) + 1, determine the new amplitude, period, phase shift and equation of the axis of the curve.

(G) Writing Sinusoidal Equations

- ex 1. Given the equation y = 2sin3(x - 60°) + 1, determine the new amplitude, period, phase shift and equation of the axis of the curve.
- Amplitude is obviously 2
- Period is 2π/3 or 360°/3 = 120°
- The equation of the equilibrium axis is y = 1
- The phase shift is 60° to the right

(G) Writing Sinusoidal Equations

- ex 2. Given a cosine curve with an amplitude of 2, a period of 180°, an equilibrium axis at y = -3 and a phase shift of 45° right, write its equation.
- So the equation is y = 2 cos [2(x - 45°)] – 3
- Recall that the k value is determined by the equation period = 2π/k or k = 2π/period
- If working in degrees, the equation is modified to period = 360°/k or k = 360°/period

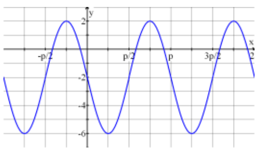
(G) Writing Sinusoidal Equations

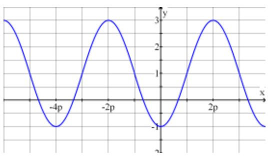
- ex 3. Write an equation and then graph each curve from the info on the table below:

	A	Period	PS	Equil
Sin	7	3π	¼ π right	-6
Cos	8	180°	None	+2
Sin	1	720°	180° right	+3
Cos	10	½ π	π left	none

(G) Writing Sinusoidal Equations

- ex 4. Given several curves, repeat the same exercise of equation writing ➔ write both a sine and a cosine equation for each graph





(H) Writing Sinusoidal Equations from Word Problems

- Now we shift to word problems wherein we must carry out the same skills in order to generate an equation for the sinusoidal curve that best models the situation being presented.
- ex 5. A small windmill has its center 6 m above the ground and the blades are 2 m in length. In a steady wind, one blade makes a rotation in 12 sec. Use the point *P* as a reference point on a blade that started at the highest point above the ground.
- (a) Determine an equation of the function that relates the height of a tip of a blade, *h* in meters, above the ground at a time *t*.
- (b) What is the height of the point *P* at the tip of a blade at 5s? 40s?
- (c) At what time is the point *P* exactly 7 m above the ground?

(H) Writing Sinusoidal Equations from Word Problems

- ex 6. In the Bay of Fundy, the depth of water around a dock changes from low tide around 03:00 to high tide at 09:00. The data shown below shows the water depth in a 24 hour period

Time (h)	0	3	6	9	12	15	18	21	24
Depth (m)	8.4	1.5	8.3	15.6	8.5	1.6	8.4	15.4	8.5

- (a) Prepare a scatter plot of the data and draw the curve of best fit
- (b) Determine an equation of the curve of best fit
- (c) You can enter the data into a GC and do a SinReg to determine the curve of best fit
- (d) Compare your equation to the calculator's equation.
- (e) Will it be safe for a boat to enter the harbour between 15:00 and 16:00 if it requires at least 3.5 m of water? Explain and confirm will algebraic calculation.

(H) Writing Sinusoidal Equations from Word Problems

Look up at the moon on a clear night. Sometimes the moon is full and the night sky is bright. At other times, there is a new moon with no visible light and the sky is dark. The moon is said to wax from dark to bright and wane back to dark.

Fraction of the Moon Visible at Midnight
Days 1 to 66 of the Year 2000

Day of the Year	1	2	3	4	5	6	10	15	20	21
Fraction of Moon Visible	0.25	0.18	0.11	0.06	0.02	0.00	0.11	0.57	0.99	1.00

Day of the Year	25	30	35	40	45	50	55	60	65	66
Fraction of Moon Visible	0.80	0.32	0.02	0.14	0.64	1.00	0.77	0.31	0.01	0.00

(H) Writing Sinusoidal Equations from Word Problems

- You found that this data represents a periodic phenomenon with the following properties:
- The period is about 29.5 days.
- The "full" moon is fully visible when the maximum value is 1.0.
- The "new" moon is not visible when the minimum value is 0.
- The axis of the curve is the horizontal line *y* 0.5.
- The amplitude of the curve is 0.5.
- You know that a sinusoidal model of this data is:
- f(x) = ***a***sin(***k***(x+***c***)) + ***d***

(H) Writing Sinusoidal Equations from Word Problems

- **1.** Draw and label a scatter plot of the data. Then draw the curve of best fit.
- **2. (a)** Starting with day 1, how many days does it take for the shortest complete pattern of the graph to repeat?
- **(b)** Starting with day 6, how many days does the graph take to repeat?
- **(c)** On what other day could the graph begin and still repeat?
- **3. (a)** Extend the pattern of the graph to include the 95th day of the new millennium. Was the phase of the moon closer to a full moon or a new moon? Explain.
- **(b)** Extend the graph to predict the fraction of the moon that was visible on the summer solstice, June 21. Was the moon waxing or waning? Explain.

(I) Homework

- Nelson text,
- Section 5.2, p420, Q1-9eol, 11-15
- Section 5.3, p433, Q1-3, 13,14,20,21,22,24,25
- Nelson text, page 464, Q8,9,10,12,13-19

(C) Angles in Standard Position – Interactive Applet

- Go to the link below and work through the ideas presented so far with respect to angles in standard position
- Angles In Trigonometry from AnalyzeMath