Lesson 38 - Solving Exponential & Logarithmic Equations

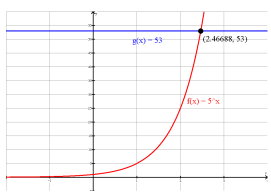
Math 2 Honors - Santowski

(A) Strategies for Solving Exponential Equations - Guessing

- we have explored a variety of equation solving strategies, namely being able to isolate a variable
- this becomes seemingly impossible for exponential equations like $5^x = 53$
- our earlier strategy was to express both sides of an equation with a common base, (i.e. $2^x = 32$) which we cannot do with the number 53 and the base of 5
- Alternatively, we can simply "guess & check" to find the right exponent on 5 that gives us 53 ➔ we know that $5^2 = 25$ and $5^3 = 125$, so the solution should be somewhere closer to 2 than 3

(B) Strategies for Solving Exponential Equations - Graphing

- Going back the example of $5^x = 53$, we always have the graphing option
- We simply graph $y_1 = 5^x$ and simultaneously graph $y_2 = 53$ and look for an intersection point (2.46688, 53)



(C) Strategies for Solving Exponential Equations - Inverses

- However, one general strategy that we have used previously was to use an "inverse" operation to isolate a variable
- and so now that we know how to "inverse" an exponential expression using logarithms, we will use the same strategy ➔ inverse an exponential using logarithms
- So then if $5^x = 53$, then $\log_5(53) = x$ ➔ but this puts us in the same dilemma as before ➔ we don't know the exponent on 5 that gives 53

(D) Strategies for Solving Exponential Equations - Logarithms

- So we will use the logarithm concept as we apply another logarithm rule ➔ let's simply take a common logarithm of each side of the equation ($\log_{10}$) (since our calculators are programmed to work in base 10)
- Thus, $5^x = 53$ now becomes
- $\log_{10}(5^x) = \log_{10}(53)$
- $\log_{10}(5)^x = \log_{10}(53)$
- $x[\log_{10}(5)] = \log_{10}(53)$ (using log rules)
- $x = \log_{10}(53) \div \log_{10}(5)$
- $x = 2.46688$ …..

(D) Strategies for Solving Exponential Equations – Natural Logarithms

- In solving $5^x = 53$, we used a common logarithm (log base 10) to solve the equation
- One other common logarithm you will see on your calculator is the natural logarithm (In which uses a special base of numerical value 2.71828… which is notated by the letter *e* ➔ so loge(x) = ln(x))
- Thus, $\ln 5^x = \ln 53$
- And $x(\ln 5) = \ln 53$
- And $x = \ln 53 \div \ln 5 = 2.46688$ as before

(E) Examples

- Evaluate $\log_3 38 = x$
- Again, same basic problem ➔ we are using a base in which 38 is an awkward number to work with (unlike 9,27,81,243,729……)
- So let's change the expression to an exponential equation ➔ $3^x = 38$ and this puts us back to the point we were at before with $5^x = 53$!!
- Thus, $\log_{10}(3)^x = \log_{10}(38)$
- And xlog3 = log38
- So x = log38 ÷ log3 = 3.31107…..

(E) Examples – Exponential Equations

- Solve the following for x
- (a) $2^x = 8$ (b) $2^x = 1.6$
- (c) $2^x = 11$ (d) $2^x = 12$
- (e) $2^{4x+1} = 8^{1-x}$ (f) $2^{x^2-4} = 8^x$
- (g) $2^{3x+2} = 9$ ` (h) $3(2^{2x-1}) = 4^{-x}$
- (i) $2^{4y+1} - 3^y = 0$
- (j) Determine the exact values of *x* which satisfy the equation:
- $64^x - 5(8^x) + 4 = 0$

(E) Examples – Log Equations

- Solve the following for x and STATE restrictions for x (WHY). Verify your solutions:
- (a) $\log_2 x = 3$ (b) $\log_2(x + 1) = 3$
- (c) ln10 – ln(7 – x) = lnx (d) 2logx – log4 = 3
- (e) 3 log(2*x* − 1) = 1
- (f) $\log_2 x + \log_2 7 = \log_2 21$
- (g) $\log_5(2x + 4) = 2$

(E) Examples – Log Equations

- (h) $2\log_9\sqrt{x} - \log_9(6x - 1) = 0$
- (i) log(x) + log(x – 1) = log(3x + 12)
- (j) $\log_2(x) + \log_2(x-2) = 3$
- (k) $\log_2(x^2 - 6x) = 3 + \log_2(1-x)$
- (l) log(x) = 1 – log(x – 3)
- (m) $\log_7(2x + 2) - \log_7(x - 1) = \log_7(x + 1)$

(F) Applications of Exponential Equations

- The half-life of radium-226 is 1620 years. After how many years is only 30 mg left if the original sample contained 150 mg?
- Recall the formula for half-life is $N(t) = N_0(2)^{(-t/h)}$ where h refers to the half-life of the substance

(F) Applications of Exponential Equations

- The half-life of radium-226 is 1620 years. After how many years is only 30 mg left if the original sample contained 150 mg?
- Recall the formula for half-life is $N(t) = N_0(2)^{(-t/h)}$ where h refers to the half-life of the substance
- or $N(t) = N_0(1+r)^t$ where r is the rate of change (or common ratio of -0.5; and t would refer to the number of "conversion periods – or the number of halving periods)
- Therefore, $30 = 150(2)^{(-t/1620)}$
- $30/150 = 0.20 = 2^{(-t/1620)}$
- log(0.2) = (-t/1620) log (2)
- log (0.2) ÷ log (2) = -t/1620
- -1620 x log(0.2) ÷ log (2) = t
- Thus t = 3761.5 years

(F) Applications of Exponential Equations

- Two populations of bacteria are growing at different rates. Their populations at time *t* are given by $P_1(t) = 5^{t+2}$ and $P_2(t) = e^{2t}$ respectively. At what time are the populations the same?

(F) Applications of Exponential Equations

- The logarithmic function has applications for solving everyday situations:
- ex 1. Mr. S. drinks a cup of coffee at 9:45 am and his coffee contains 150 mg of caffeine. Since the half-life of caffeine for an average adult is 5.5 hours, determine how much caffeine is in Mr. S.'s body at class-time (1:10pm). Then determine how much time passes before I have 30 mg of caffeine in my body.
- ex 2. The value of the Canadian dollar , at a time of inflation, decreases by 10% each year. What is the half-life of the Canadian dollar?

(F) Applications of Exponential Equations

- ex. Find the length of time required for an investment of $1000 to grow to $4,500 at a rate of 9% p.a. compounded quarterly.
- Bakersfield, California was founded in 1859 when Colonel Thomas Baker planted ten acres of alfalfa for travellers going from Visalia to Los Angeles to feed their animals. The city's population can be modelled by the equation $P(t) = 33{,}400e^{0.0397t}$, where *t* is the number of years since 1950.
  - a. Has Bakersfield experienced growth or decline in population? Explain.
  - b. What was Bakersfield's population in 1950?
  - c. Find the projected population of Bakersfield in 2010.

(F) Applications of Exponential Equations

- The value, $V, of a particular car can be modelled by the equation $V(t) = ke^{-\lambda t}$ where *t* years is the age of the car. The car's original price was $7499, and after one year it is valued at $6000. State the value of *k* and calculate λ giving your answer to 2 decimal places. Hence obtain the value of the car when it is three years old.

(F) Applications of Exponential Equations

- ex 5. Dry cleaners use a cleaning fluid that is purified by evaporation and condensation after each cleaning cycle. Every time it is purified, 2% of the fluid is lost
- (a) An equipment manufacturer claims that after 20 cycles, about two-thirds of the fluid remains. Verify or reject this claim.
- (b) If the fluid has to be "topped up" when half the original amount remains, after how many cycles should the fluid be topped up?
- (c) A manufacturer has developed a new process such that two-thirds of the cleaning fluid remains after 40 cycles. What percentage of fluid is lost after each cycle?

(G) Internet Links

- College Algebra Tutorial on Exponential Equations (NOTE: this lesson uses natural logarithms to solve exponential equations)
- Solving Exponential Equations Lesson from Purple Math (NOTE: this lesson uses natural logarithms to solve exponential equations)
- SOLVING EXPONENTIAL EQUATIONS from SOS Math (NOTE: this lesson uses natural logarithms to solve exponential equations)

(H) Homework

- p. 407 # 9-25 odds, 26, 27, 30-33