Lesson 21 – Roots of Polynomial Functions

Math 2 Honors - Santowski

Lesson Objectives

- Reinforce the understanding of the connection between factors and roots
- Mastery of the factoring of polynomials using the algebraic processes of long & synthetic division & various theorems like RRT, RT & FT
- Introduce the term "multiplicity of roots" and illustrate its graphic significance
- Solve polynomial equations for x being an element of the set of complex numbers
- State the Fundamental Theorem of Algebra

(A) Multiplicity of Roots

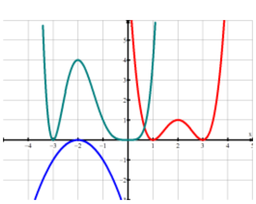
- Factor the following polynomials:
- $P(x) = x^2 – 2x – 15$
- $P(x) = x^2 – 14x + 49$
- $P(x) = x^3 + 3x^2 + 3x + 1$
- Now solve each polynomial equation, P(x) = 0
- Solve $0 = 5(x + 1)^2(x – 2)^3$
- Solve $0 = x^4(x – 3)^2(x + 5)$
- Solve $0 = (x + 1)^3(x – 1)^2(x – 5)(x + 4)$

(A) Multiplicity of Roots

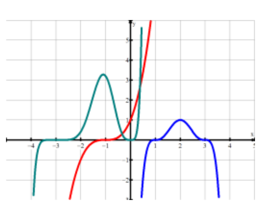
- If *r* is a zero of a polynomial and the exponent on the factor that produced the root is *k*, $(x – r)^k$, then we say that *r* has **multiplicity** of *k*. Zeroes with a multiplicity of 1 are often called **simple** zeroes.
- For example, the polynomial $x^2 – 14x + 49$ will have one zero, x = 7, and its multiplicity is 2. In some way we can think of this zero as occurring twice in the list of all zeroes since we could write the polynomial as, $(x – 7)^2 = (x – 7)(x – 7)$
- Written this way the term (x – 7) shows up twice and each term gives the same zero, x = 7.
- Saying that the multiplicity of a zero is *k* is just a shorthand to acknowledge that the zero will occur *k* times in the list of all zeroes.

(A) Multiplicity ➔ Graphic Connection

Even Multiplicity



Odd Multiplicity



(B) Solving if x ε **C**

- Let's expand our number set from real numbers to complex numbers
- Factor and solve $3 – 2x^2 – x^4 = 0$ if x ε **C**
- Factor and solve $3x^3 – 7x^2 + 8x – 2 = 0$ if x ε **C**
- Factor and solve $2x^3 + 14x - 20 = 9x^2 – 5$ if x ε **C**
- Now write each polynomial as a product of its factors
- Explain the graphic significance of your solutions for x

(B) Solving if x ε **C** – Solution to Ex 1

- Factor and solve $3 - 2x^2 - x^4 = 0$ if x ε **C** and then write each polynomial as a product of its factors
- Solutions are $x = \pm 1$ and $x = \pm i\sqrt{3}$
- So rewriting the polynomial in factored form (over the reals) is $P(x) = -(x^2 + 3)(x - 1)(x + 1)$ and over the complex numbers: $P(x) = -(x-1)(x+1)(x-i\sqrt{3})(x+i\sqrt{3})$

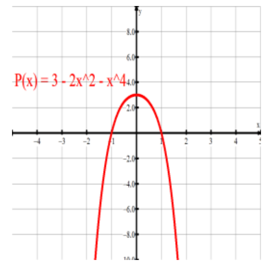
(B) Solving if x ε **C** – Graphic Connection

- With $P(x) = 3 - 2x^2 - x^4$, we can now consider a graphic connection, given that

  $P(x) = -(x^2 + 3)(x - 1)(x + 1)$

  or given that

  $P(x) = -(x-1)(x+1)(x-i\sqrt{3})(x+i\sqrt{3})$

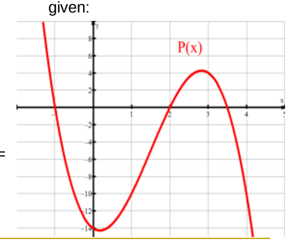


(C) Fundamental Theorem of Algebra

- The fundamental theorem of algebra can be stated in many ways:
  - (a) If P(x) is a polynomial of degree *n* then P(x) will have exactly *n* zeroes (real or complex), some of which may repeat.
  - (b) Every polynomial function of degree $n \geq 1$ has exactly *n* complex zeroes, counting multiplicities
  - (c) If P(x) has a nonreal root, $a+b\mathbf{i}$, where $b \neq 0$, then its conjugate, $a-b\mathbf{i}$ is also a root
  - (d) Every polynomial can be factored (over the real numbers) into a product of linear factors and irreducible quadratic factors
- What does it all mean → we can solve EVERY polynomial (it may be REALLY difficult, but it can be done!)

(D) Using the FTA

- Write an equation of a polynomial whose roots are x = 1, x = 2 and x = ¾
- Write the equation of the polynomial whose roots are 1, -2, -4, & 6 and a point (-1, -84)
- Write the equation of a polynomial whose roots are x = 2 (with a multiplicity of 2) as well as $x = -1 \pm \sqrt{2}$
- Write the equation of a polynomial whose graph is given:



(D) Using the FTA

- Given that 1 – 3i is a root of $x^4 - 4x^3 + 13x^2 - 18x - 10 = 0$, find the remaining roots.
- Write an equation of a third degree polynomial whose given roots are 1 and *i*. Additionally, the polynomial passes through (0,5)
- Write the equation of a quartic wherein you know that one root is 2 – *i* and that the root x = 3 has a multiplicity of 2.

(E) Further Examples

- The equation $x^3 - 3x^2 - 10x + 24 = 0$ has roots of 2, *h*, and *k*. Determine a quadratic equation whose roots are *h* – *k* and *hk*.
- The 5th degree polynomial, f(x), is divisible by x^3 and f(x) – 1 is divisible by $(x - 1)^3$. Find f(x).
- Find the polynomial p(x) with integer coefficients such that one solution of the equation p(x)=0 is $1+\sqrt{2}+\sqrt{3}$.

(E) Further Examples

- Start with the linear polynomial: $y = -3x + 9$. The x-coefficient, the root and the intercept are -3, 3 and 9 respectively, and these are in arithmetic progression. Are there any other linear polynomials that enjoy this property?
- What about quadratic polynomials? That is, if the polynomial $y = ax^2 + bx + c$ has roots r_1 and r_2 can a, r_1, b, r_2 and c be in arithmetic progression?

Homework

- Textbook, S7.5, p463-464, Q17,19,27,28,31,32,38,43,45,46,48,49,50
- Do some with & some without the TI-84