Lesson 19 – Factoring Polynomials

Math 2 Honors - Santowski

Fast Five

- Factor the number 38,754 (NO CALCULATOR)
- Divide 72,765 by 38 (NO CALCULATOR)
- How would you know if 145 was a factor of 14,436,705? What about 215?

Lesson Objectives

- Understand the similarities and differences between the algebraic processes involving second degree polynomials (quadratics) and higher order polynomials (cubics, quartics)
- Understand that factoring involves division and the connections between the 2 processes
- Mastery of the factoring of polynomials using the algebraic processes of long & synthetic division
- Introduce the remainder and rational root theorems and apply them to factor polynomials
- Reinforce the understanding of the connection between factors and roots

(A) Why do algebra with polynomials?

- In our work with quadratic polynomials, we focused on the algebraic analysis of quadratics ➔ WHY ➔ we were interested in finding either (i) points on the parabola, (ii) the extrema (vertex) or (iii) zeroes, x-intercepts, roots
- What we saw was that quadratics can be presented in three forms (standard, vertex, factored) ➔ from which we can perform "simple" algebra in order to determine the (i) points on the parabola, (ii) the extrema (vertex) or (iii) zeroes, x-intercepts, roots

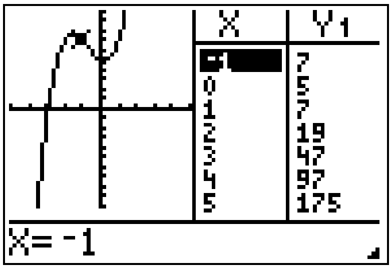
(A) Why do algebra with polynomials?

- But now that we are consider further polynomials (cubics, quartics) ➔ do the same "simple" algebraic processes apply??? ➔ NO!!
- Through "simple" algebraic processes with polynomials, we can only "easily" determine the (i) points on the parabola, and (ii) zeroes, x-intercepts, roots
- To find the extrema (max,min), we need other algebraic processes ➔ CALCULUS

(B) Algebra of Polynomials ➔ Evaluation of Polynomials

- It is relatively easy to EVALUATE any function ➔ simply substitute in a value for the variable
- i.e if $P(x) = x^3 + 2x^2 – x + 5$, evaluate $P(-1)$
- so $P(-1) = (-1)^3 + 2(-1)^2 – (-1) + 5 = -1 + 2 + 1 + 5 = 7$
- So $P(-1) = 7$ ➔ GRAPHIC/NUMERIC connection ➔ it means that we have the order pair/point (-1,7) on our table of values and on our graph

(B) Algebra of Polynomials ➔ Evaluation of Polynomials ➔ Graphic & Numeric Connection



(B) Algebra of Polynomials ➔ Evaluation of Polynomials ➔ examples

- Evaluate $4x^3 - 6x^2 + x - 3$ for $x = 2$
- Evaluate $P(-4)$ if $P(x) = 2x^3 + 16x^2 + 23x - 36$ (significance?????)
- Evaluate $x^2 + 6x^3 - 5$ for $x = ½$
- Evaluate $P(2)$ if $P(x) = x^4 + 4x^3 + 2x^2 - 3x - 50$

(C) Roots & Factors

- In our work with quadratics, we saw the "factored" form or "intercept" form of a quadratic equation/expression
- i.e. $f(x) = x^2 – x – 2 = (x – 2)(x + 1)$ ➔ factored form of eqn
- So when we solve $f(x) = 0$ ➔ $0 = (x – 2)(x + 1)$, we saw that the zeroes/x-intercepts/roots were $x = 2$ and $x = -1$
- So we established the following connection:
- Factors ➔ $(x - 2)$ and $(x – 1)$
- Roots ➔ $x = 2$ and $x = 1$
- So we will now reiterate the following connections:
- If $(x – R)$ is a ***factor*** of $P(x)$, then $x = R$ is ***root*** of $P(x)$
  AND THE CONVERSE
- If $x = R$ is a ***root*** of $P(x)$, then $(x – R)$ is a ***factor*** of $P(x)$

(D) Factoring ➔ The Process

- So how do we factor in general?
- Factor the number 38,754 ➔ (2)(3)(3)(2153)
- KEY IDEA ➔ the PROCESS that we used was DIVISION

(E) Division of Polynomials

- So, in order to factor, we need to be able to DIVIDE
- Q ➔ HOW do we divide polynomials??
- We will show you 2 ways to divide polynomials ➔ long division & synthetic division
- Keep in mind WHY we are factoring the polynomials ➔ I will eventually ask for ROOTS

(E) Division of Polynomials

- Divide using LD
- $3x^3 + 13x^2 - 9x + 6$ by $x + 5$

$$x+5\overline{)3x^3+13x^2-9x+6} \quad ????????$$

- Divide using SD:
- $3x^3 + 13x^2 - 9x + 6$ by $x + 5$

-5| 3 13 -9 6

_______________.

(F) Practice - Examples

- Show examples 1,2,3 using both division methods
- ex 1. Divide $x^3 - 42x + 30$ by $x - 6$
- ex 2. Divide $x^2 + 6x^3 - 5$ by $2x - 1$
- ex 3. Divide $x^4 + 4x^3 + 2x^2 - 3x - 50$ by $x - 2$

(G) Division & Factoring Connection

- Recall ➔ WHY are we dividing?
- ANS ➔ to see if the divisor is a FACTOR of the dividend
- WHAT do we look for when we do the division to answer the "factor" question ➔ the remainder!!
- Is 4 a factor of 72? How do you know??
- Is 5 a factor of 72? How do you know??

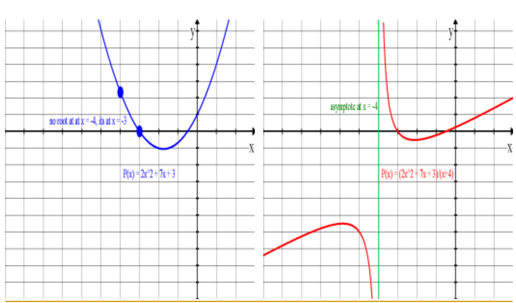
(H) Factoring Polynomials – An Example

- So we have in place a method for factoring polynomials ➔ use division to see if the remainder is 0!!
- Is (x + 4) a factor of $P(x) = 2x^2 + 7x + 3$

(H) Factoring Polynomials – An Example

- When dividing $(2x^2 + 7x + 3)$ by $(x + 4)$ and we get $2x - 1$ with a remainder of 4
- *Conclusions to be made*
- (i) $x + 4$ is a not factor of $2x^2 + 7x + 3$
- (ii) $x + 4$ does not divide evenly into $2x^2 + 7x + 3$
- (iii) when $2x^2 + 7x + 3$ is divided by $x + 4$, the remainder is 7
- (iv) $2x^2 + 7x + 3 = (x + 4)(2x - 1) + 7$
- (v) $(2x^2 + 7x + 3)/(x + 4) = 2x - 1 + 7/(x + 4)$
- *Show on GC and make connections*
- (i) graph $2x^2 + 7x + 3$ and see that $x = -4$ is not a root
- (ii) graph $(2x^2 + 7x + 3)/(x + 4)$ ➔ and we see???

(H) Factoring Polynomials – An Example



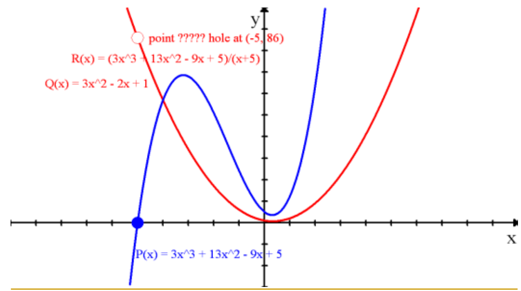
(H) Factoring Polynomials – An Example

- So we have in place a method for factoring polynomials ➔ use division to see if the remainder is 0!!
- Q? Is x + 5 a factor of $3x^3 + 13x^2 - 9x + 6$??
- How do you know?
- Q? Is x + 5 a factor of $3x^3 + 13x^2 - 9x + 5$??
- How do you know?

(H) Factoring Polynomials – An Example

- Divide $3x^3 + 13x^2 - 9x + 5$ by $x + 5$
- *conclusions to be made:* - all 5 conclusions are equivalent and say mean the same thing
- (i) $x + 5$ is a factor of $3x^3 + 13x^2 - 9x + 5$
- (ii) $x + 5$ divides evenly into $3x^3 + 13x^2 - 9x + 5$
- (iii) when $3x^3 + 13x^2 - 9x + 5$ is divided by $x + 5$, there is no remainder
- (iv) $3x^3 + 13x^2 - 9x + 5 = (x + 5)(3x^2 - 2x + 1)$
- (v) $(3x^3 + 13x^2 - 9x + 5)/(x + 5) = 3x^2 - 2x + 1$
- *Show on GC and make connections*
- i) graph $3x^3 + 13x^2 - 9x + 5$ and see that $x = -5$ is a root or a zero or an x-intercept
- ii) graph $(3x^3 + 13x^2 - 9x + 5)/(x + 5)$ ➔ and we see??

(H) Factoring Polynomials – An Example



(I) Factoring Polynomials – Practice

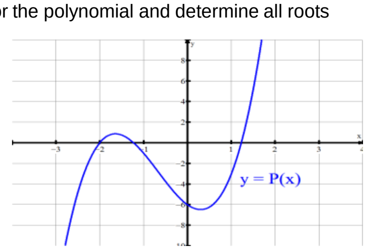
- Factor $g(x) = x^3 + 2x^2 - 16x - 32$
- Factor $P(x) = x^3 - 3x^2 - 2x + 6$
- Factor $f(x) = x^3 – x^2 - 14x + 24$
- Factor $P(x)$ $x^3 – 6x^2 + 13x – 20$
- Factor $y = x^3 + 4x^2 + 7x + 6$
- Factor $y = x^3 – 9x^2 + 24x – 16$
- Factor $f(x) = x^4 + x^3 – 11x^2 – 9x + 18$
- Factor $g(x) = x^4 – 3x^3 + 6x^2 – 2x - 12$

(I) Factoring Polynomials – Practice

- Special Polynomials ➔ just like we saw perfect square trinomials and difference of square trinomials in our quadratics chapter, we have similar "special" cubics
- Factor $x^3 – 8$
- Factor $27x^3 - 64$
- Now what would you generalize for $(ax)^3 – b^3$?
- Factor $x^3 + 8$
- Factor $27x^3 + 64$
- Now what would you generalize for $(ax)^3 + b^3$?

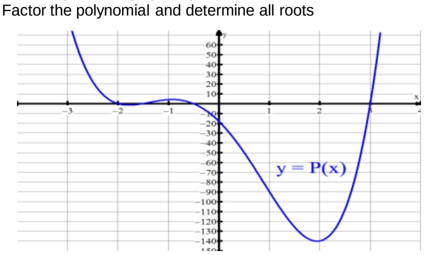
(I) Factoring Polynomials – Practice

- You are given the graph of $y = 2x^3 + 4x^2 – 3x – 6$. Factor the polynomial and determine all roots



(I) Factoring Polynomials – Practice

- You are given the graph of $y = 4x^4 + 4x^3 – 29x^2 – 51x – 18$. Factor the polynomial and determine all roots



(J) Factoring Polynomials – The Remainder Theorem

- So in factoring P(x), we use the remainder of the division in order to make a decision about whether or not the x – A is/is not a factor of P(x)
- So if we only want to find a remainder, is there another way (rather than only division?)
- YES there is ➔ it's called the Remainder Theorem

(J) Factoring Polynomials – The Remainder Theorem

- Divide $-2x^3 + 3x^2 - 39x - 20$ by x + 1
- Evaluate P(-1). What do you notice?
- What must be true about x + 1

- Divide $x^3 – 8x^2 + 11x + 5$ by x - 2
- Evaluate P(2). What do you notice?
- What must be true about (x – 2)?

- Divide $3x^3 – 4x^2 - 2x - 5$ by x + 1
- Evaluate P(-1). What do you notice?

- EXPLAIN WHY THIS WORKS????

(J) Factoring Polynomials – The Remainder Theorem

- Divide $3x^3 – 4x^2 - 2x - 5$ by x + 1
- $Q(x) = (3x^2 – 7x + 5)$ with a remainder of -10
- Evaluate P(-1) ➔ equals -10

- EXPLAIN WHY THIS WORKS????

- if rewritten as:
- $3x^3 – 4x^2 - 2x - 5 = (x + 1)(3x^2 - 7x + 5) - 10$,
- $P(-1) = (-1 + 1)(3(-1)^2 – 7(-1) + 5) – 10 = -10$

(J) Factoring Polynomials – The Remainder Theorem

- the remainder theorem states "when a polynomial, P(x), is divided by (ax - b), and the remainder contains *no term in x*, then the remainder is equal to P(b/a)
- PROVE WHY THIS IS TRUE ?!?!?!?!?

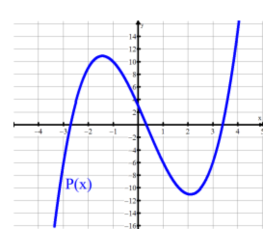
(K) Factoring Polynomials – The Remainder Theorem - Examples

- Factor the following polynomials using the RRT and the Remainder Theorem:

- $P(x) = -2x^3 – x^2 + 25x – 12$
- $P(x) = 4x^3 – 12x^2 – 19x + 12$
- $P(x) = 12x^4 + 32x^3 – 15x^2 – 8x + 3$

(K) Factoring Polynomials – The Remainder Theorem - Connection

- So, from the graph on the right, determine the remainder when P(x) is divided by:
- (i) x + 2
- (ii) x – 2
- (iii) x – 4



(K) Factoring Polynomials – The Remainder Theorem - Examples

- Find *k* so that when $x^2 + 8x + k$ is divided by $x - 2$, the remainder is 3
- Find the value of *k* so that when $x^3 + 5x^2 + 6x + 11$ is divided by $x + k$, the remainder is 3
- When $P(x) = ax^3 – x^2 - x + b$ is divided by $x - 1$, the remainder is 6. When P(x) is divided by $x + 2$, the remainder is 9. What are the values of *a* and *b*?
- Use the remainder theorem to determine if $(x – 4)$ is a factor of P(x) if $P(x) = x^4 – 16x^2 – 2x + 6$

(L) Factoring Polynomials – the Rational Root Theorem

- Our previous examples were slightly misleading … as in too easy (leading coefficient was deliberately 1)
- Consider this example ➔ $-12x^3 + 20x^2 + 33x – 20$ which when factored becomes (2x-1)(3x+4)(5-2x) so the roots would be ½, -4/3, and 5/2
- Make the following observation ➔ that the numerator of the roots (1, -4, 5) are factors of the constant term (-20) while the denominator of the roots (2,3,2) are factors of the leading coefficient (-12)
- We can test this idea with other polynomials ➔ we will find the same pattern ➔ that the roots are in fact some combination of the factors of the leading coefficient and the constant term

(L) Factoring Polynomials – the Rational Root Theorem

- Our previous observation (although limited in development) leads to the following theorem:
- Given $P(x) = a_nx^n + a_{n-1}x^{n-1} + ….. + a_1x^1 + a_0$, if P(x) = 0 has a rational root of the form *a/b* and *a/b* is in lowest terms, then *a* must be a divisor of a_0 and *b* must be a divisor of a_n

(L) Factoring Polynomials – the Rational Root Theorem

- So what does this theorem mean?
- If we want to factor $P(x) = 2x^3 – 5x^2 + 22x – 10$, then we first need to find a value *a/b* such that P(*a/b*) = 0
- So the factors of the leading coefficient are {±1,±2} which are then the possible values for *a*
- The factors of the constant term, -10, are {±1,±2,±5,±10} which are then the possible values for *b*
- Thus the possible ratios *a/b* which we can test to find the factors are {±1,±½ ,±2,±5/2,±5,±10}
- As it then turns out, P(½) turns out to give P(x) = 0, meaning that (x – ½) or (2x – 1) is a factor of P(x)

(M) Factoring Polynomials – the Rational Root Theorem - Examples

- Ex 1. To factor $P(x) = 2x^3 – 9x^2 + 7x + 6$, what values of x could you test according to the RRT
- Now factor P(x)
- Ex 2. To factor $P(x) = 3x^3 – 7x^2 + 8x – 2$ what values of x could you test according to the RRT
- Now factor P(x)
- ex 3 ➔ Graph $f(x) = 3x^3 + x^2 - 22x - 24$ using intercepts, points, and end behaviour. Approximate turning points, max/min points, and intervals of increase and decrease.

(N) Conclusion

- So, we now have some “simple” algebra tools that we can use to factor polynomials
- We use the rational root theorem and the remainder theorem
- We use these techniques in order to determine whether a chosen binomial (ax + b) is or is not a factor of our polynomial

Homework

- Homework:
- Ex. 7.3 (p. 429) # 26-56 evens, 62-65, 73-83 odds, 87, 93, 97, 99