Lesson 16 - Quadratic Equations & Complex Numbers

Math 2 Honors - Santowski

Lesson Objectives

- Find and classify all real and complex roots of a quadratic equation
- Understand the "need for" an additional number system
- Add, subtract, multiply, divide, and graph complex numbers
- Find and graph the conjugate of a complex number

Fast Five

- STORY TIME.....
- http://mathforum.org/johnandbetty/frame.htm

(A) Introduction to Complex Numbers

- Solve the equation $x^2 - 1 = 0$
- We can solve this many ways (factoring, quadratic formula, completing the square & graphically)
- In all methods, we come up with the solution $x = \pm 1$, meaning that the graph of the quadratic (the parabola) has 2 roots at $x = \pm 1$.
- Now solve the equation $x^2 + 1 = 0$

(A) Introduction to Complex Numbers

- Now solve the equation $x^2 + 1 = 0$
- The equation $x^2 = -1$ has no roots because you cannot take the square root of a negative number.
- Long ago mathematicians decided that this was too restrictive.
- They did not like the idea of an equation having no solutions -- so they invented them.
- They proved to be very useful, even in practical subjects like engineering.

(A) Introduction to Complex Numbers

- Consider the general quadratic equation $ax^2 + bx + c = 0$ where $a \neq 0$.
- The usual formula obtained by ``completing the square'' gives the solutions $$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
- If $b^2 \geq 4ac$ (or if $b^2 - 4ac \geq 0$) we are "happy".

(A) Introduction to Complex Numbers

- If $b^2 \geq 4ac$ (or if $b^2 - 4ac \geq 0$) we are happy.
- If $b^2 < 4ac$ (or if $b^2 - 4ac < 0$) then the number under the square root is negative and you would say that the equation has no solutions.
- In this case we write $b^2 - 4ac = (-1)(4ac - b^2)$ and $4ac - b^2 > 0$. So, in an obvious formal sense, $$x = \frac{-b \pm \sqrt{-1}\sqrt{4ac - b^2}}{2a}$$
- and now the only `meaningless' part of the formula is $\sqrt{-1}$

(A) Introduction to Complex Numbers

- So we might say that any quadratic equation either has ``real'' roots in the usual sense or else has roots of the form $p + q\sqrt{-1}$ where p and q belong to the real number system .
- The expressions $p + q\sqrt{-1}$ do not make any sense as real numbers, but there is nothing to stop us from playing around with them as <u>**symbols**</u> as $p + qi$ (but we will use $a + bi$)
- We call these numbers complex numbers; the special number i is called an imaginary number, even though i is just as ``real'' as the real numbers and complex numbers are probably simpler in many ways than real numbers.

(B) Using Complex Numbers → Solving Equations

- Note the difference (in terms of the expected solutions) between the following 2 questions:

- Solve $x^2 + 2x + 5 = 0$ where $x \in R$
- Solve $x^2 + 2x + 5 = 0$ where $x \in C$

(B) Using Complex Numbers → Solving Equations

- Solve the following quadratic equations where $x \in C$
- Simplify all solutions as much as possible

- $x^2 - 2x = -10$
- $3x^2 + 3 = 2x$
- $5x = 3x^2 + 8$
- $x^2 - 4x + 29 = 0$

- What would the "solutions" of these equations look like if $x \in R$

(C) Operations with Complex Numbers

- So if we are going to "invent" a number system to help us with our equation solving, what are some of the properties of these "complex" numbers?

- How do we operate (add, sub, multiply, divide)
- How do we graphically "visualize" them?
- Powers of i
- Absolute value of complex numbers

(D) Adding/Subtracting Complex Numbers

- Property of real numbers → Closure

- Q? Is closure a property of complex numbers?
- Well, lets see HOW to add/subtract complex numbers!

- to add or subtract two complex numbers, $z_1 = a + ib$ and $z_2 = c+id$, the rule is to add the real and imaginary parts separately:

- $z_1 + z_2 = a + ib + c + id = a + c + i(b + d)$
- $z_1 - z_2 = a + ib - c - id = a - c + i(b - d)$

- Example
- (a) $(1 + i) + (3 + i) = 1 + 3 + i(1 + 1) = 4 + 2i$
- (b) $(2 + 5i) - (1 - 4i) = 2 + 5i - 1 + 4i = 1 + 9i$

(D) Adding/Subtracting Complex Numbers

- Exercise 1. Add or subtract the following complex numbers.

- (a) $(3 + 2i) + (3 + i)$
- (b) $(4 - 2i) - (3 - 2i)$
- (c) $(-1 + 3i) + (2 + 2i)$
- (d) $(2 - 5i) - (8 - 2i)$

(D) Adding/Subtracting Complex Numbers

- Property of real numbers ➔ Commutative

- Q? Is the addition/subtraction of complex numbers commutative?

- Exercise 2. Use the following complex numbers to answer our question.

- (a) $(3 + 5i) + (4 + i)$
- (b) $(4 - 2i) - (7 - 3i)$
- (c) $(-6 + 3i) + (2 + i)$
- (d) $(2 - 5i) - (8 - 2i)$

(E) Multiplying Complex Numbers

- We multiply two complex numbers just as we would multiply expressions of the form $(x + y)$ together

- $(a + ib)(c + id) = ac + a(id) + (ib)c + (ib)(id)$
- $= ac + iad + ibc - bd$
- $= ac - bd + i(ad + bc)$

- Example
- $(2 + 3i)(3 + 2i)$
- $= 2 \times 3 + 2 \times 2i + 3i \times 3 + 3i \times 2i$
- $= 6 + 4i + 9i - 6$
- $= 13i$

(E) Multiplying Complex Numbers

- Exercise 3. Multiply the following complex numbers.

- (a) $(3 + 2i)(3 + i)$
- (b) $(4 - 2i)(3 - 2i)$
- (c) $(-1 + 3i)(2 + 2i)$
- (d) $(2 - 5i)(8 - 3i)$
- (e) $(2 - i)(3 + 4i)$

- T/F ➔ multiplication of complex numbers shows the closure & commutative property ➔ justify with an example and then PROVE it to be true/false

(F) Complex Conjugation

- For any complex number, z = a+ib, we define the complex conjugate to be: $\bar{z} = a - ib$.
- It is very useful since the following are real:
- $z + \bar{z}$ = a + ib + (a − ib) = 2a
- $z\bar{z}$ = (a + ib)(a − ib) = a^2 + iab − iab − $(ib)^2$ = $a^2 + b^2$
- The modulus of a complex number is defined as: $|z| = \sqrt{z\bar{z}}$
- Exercise 4. Combine the following complex numbers and their conjugates.
- (a) If z = (3 + 2i), find $z + \bar{z}$ (b) If z = (3 − 2i), find $z\bar{z}$
- (c) If z = (−1 + 3i), find $z\bar{z}$ (d) If z = (4 − 3i), find |z|

(G) Dividing Complex Numbers

- The trick for dividing two complex numbers is to multiply top and bottom by the complex conjugate of the denominator:

$$\frac{z_1}{z_2} = \frac{z_1 \times \bar{z}_2}{z_2 \times \bar{z}_2}$$

- Example:

- $$\frac{3-i}{2+2i} = \frac{(3-i)\times(2-2i)}{(2+2i)\times(2-2i)}$$
$$= \frac{6-2i+6i+2i^2}{4-4i+4i-4i^2}$$
$$= \frac{6+2(-1)+4i}{4-4(-1)}$$
$$= \frac{4+4i}{8}$$
$$= \frac{1+i}{2}$$

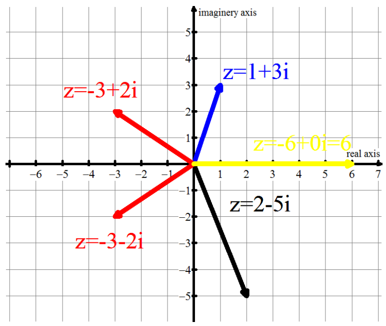
(G) Dividing Complex Numbers

- Exercise 5. Perform the following divisions:
- (a) (2 + 4i)/i
- (b)(−2 + 6i)/(1 + 2i)
- (c) (1 + 3i)/(2 + i)
- (d) (3 + 2i)/ (3 + i)

(H) Graphing Complex Numbers

- So graphing a real number is easy ➔ use a number line
- So then, where do you graph complex numbers on a REAL number line??
- You don't ➔ use "invent"/develop an alternative graphic representation of a complex number
- Since complex numbers have "two parts" to them (a real part, *a*, and a complex part, *bi*) could we use this "two parts" as a strategy for representing them graphically?

(H) Graphing Complex Numbers



(H) Graphing Complex Numbers

- Graph the following complex numbers:

- z = 3 + 2i
- z = -5 + 4i
- z = -6 – 3i
- z = 2i
- z = 5

- Show a graphic representation of vector addition wherein you work with z_1 = 3 + 5i and z_2 = -4 – 2i ➔ show $z_1 + z_2$.
- How about vector subtraction ➔ try $z_1 - z_2$ and then $z_2 - z_1$

(I) Absolute Value of Complex Numbers

- When working with real numbers, the absolute value of a number was defined as ……. ???
- So, in complex numbers, the idea is the same ➔ ……
- So, since we have just finished graphing complex numbers ➔
- Determine the value of and graph:
- (a) |2 – 3i|
- (b) |3 – 5i|
- (c) |4 + 3i|

(J) HOMEWORK

- p. 319 # 11-21 odds, 39-47 odds, 48, 52, 53-75 odds, 85-95 odds, 96-99